# Truthmakers and Normative Conflicts

Albert Anglberger
University of Bayreuth
aangl@gmx.at

Johannes Korbmacher
Utrecht University
j.korbmacher@uu.nl

March 31, 2019*

**Abstract**

By building on work by Kit Fine, we develop a sound and complete truthmaker semantics for Lou Goble's conflict tolerant deontic logic **BDL**.

## 1 Introduction

A *normative conflict* consists of a number of obligations that cannot jointly be met. Normative conflicts arise, for example, in the context of *moral dilemmas*. Think of Sartre's famous example of the French student [14]: During World War 2, the student faces a choice between two possible courses of action: he can either join the French resistance to avenge his brother, who was killed by the invading German forces, *or* he can take care of his widowed mother, of whom he is the only living relative. The student cannot do both, but, as Sartre argues, the student has an obligation to do both. He faces a normative conflict. A *conflict tolerant deontic logic* (CTDL) is a deontic logic in which normative conflicts are consistent. CTDLs are needed when one, like Sartre, holds that there are real normative conflicts, which cannot be dissolved, e.g. by showing that what seems like an obligation isn't really one. When faced with a normative conflict, one should still be able to reason about the implications of one's (conflicting) obligations, to decide what's the best course of action. A conflict *in*tolerant deontic logic, which renders normative conflicts inconsistent, would, of course, not be of much help here: since from a contradiction anything follows, in such a logic one cannot reasonably reason about the conflict at hand. This is *the problem of conflict tolerance*.

What should such a CTDL look like? In a recent survey, Lou Goble sets out to answer this question, and he proposes three desiderata a CTDL should satisfy [11]:

**Consistent Conflicts.** At least some normative conflicts should be consistent, i.e. we can have $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$ but $OA_1, \ldots, OA_n \nvdash \bot$ (p. 297).

*This is a post-peer-review, pre-copyedit version of an article published in *Studia Logica*. The final authenticated version is available online at: https://doi.org/10.1007/s11225-019-09862-5.

**No Deontic Explosion.** Normative conflicts should not result in *deontic explosion*, i.e. we can have $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$ but $OA_1, \ldots, OA_n \nvdash OB$ for some $B$ (p. 298).[1]

**Minimal Deontic Laws.** Certain minimal laws of deontic logic, which are plausible from considerations independent of any particular view of deontic conflicts, should be validated (p. 302). As examples, Goble explicitly mentions:

| | | |
|---|---|---|
| (DDS) | $O(A \vee B), O\neg A \vdash OB$ | ('deontic disjunctive syllogism'). |
| (M) | $O(A \wedge B) \vdash OB$ | ('monotonicity'). |
| (AGG) | $OA, OB \vdash O(A \wedge B)$ | ('aggregation'). |

There are different routes one might take here.[2] In this paper, we're interested in what Goble calls logics "with limited replacement", i.e. systems in which the rule of substitution can only be applied for a restricted class of statements. In particular, we're interested in Goble's system **BDL**, in which substitution is only allowed for what Goble calls *analytically* equivalent statements [11, p. 315–18].

**BDL** is a promising candidate for a CTDL since it satisfies all of Goble's desiderata. But as Goble himself points out:

> On the formal front, **BDL** [. . . ] so far lack[s] any semantics or model theory, and it is difficult to see how that might be developed, while respecting the limits necessary to protect their treatment of normative conflicts. [11, p. 318]

In this paper, we develop a sound and complete semantics for **BDL**. What makes this particularly challenging is the fact that Goble's desiderata rule out any semantics that validates replacement of classical equivalences inside the $O$ operator, such as possible world semantics (for more on that, see below).

By building on Kit Fine's recent truthmaker semantics for analytic equivalence [8], we are able to meet Goble's challenge. We propose new, conflict tolerant semantic clauses for statements of obligation, which are partially inspired by Fine's truthmaker treatment of statements of permission [7]. Based on a notion of admissible states, Kit Fine essentially proposed the following truth-condition for permission statements of the form $PA$ (for "it is permitted that $A$"):

- $PA$ is true iff every state that is a truthmaker of $A$ is admissible.

We will now use this very notion of admissible states to develop a semantics for obligation. Put intuitively, our truth-condition for obligation statements is this:

- $OA$ is true iff there is no admissible state that is a falsemaker of $A$.

[1] Note that in the presence of the principle of *Ex Contradictione Quodlibet (ECQ)*, i.e. $\perp \vdash A$ for all $A$, **No Deontic Explosion** entails **Consistent Conflicts**. The two are kept apart so as not to decide from the very start that the background logic needs to have (ECQ). The logics we'll be studying in this paper do have (ECQ) and, correspondingly, we can focus our attention on **No Deontic Explosion**. In our exposition, however, we follow Goble's paper.

[2] For an overview of these options, see §4 and §5 of [11].

In what follows, we start by developing a sound and complete semantics for the weaker system $\mathbf{BDL}^{-}$, which we'll introduced in this paper as **BDL** without (DDS). However, by adding an additional semantic constraint, we'll also be able obtain the validity of (DDS), resulting in a sound and complete semantics for full **BDL**.

The structure of the paper is as follows. In §2, we discuss the limited replacement approach to CTDLs and introduce Goble's system **BDL**. We show that Goble's proposed system for analytic equivalence is deductively equivalent to Angell's system AC of analytic containment [2]. In the following section, §3, we extend Kit Fine's truthmaker semantics for AC with our conflict tolerant semantic clauses for obligation. In §4, we give a semantic constraint on our models that is equivalent to the validity of (DDS), thereby obtaining a sound and complete semantics for **BDL**. In §5, we prove the main result of our paper: the soundness of completeness of our semantics for $\mathbf{BDL}^{-}$ and **BDL**. We conclude the paper, in §6, with a few general remarks on **BDL** and a discussion of possible concepts of permission in **BDL** and their interaction with obligation.

### Syntax

The syntax in our paper is as follows: Our *base language* $\mathcal{L}$ is a propositional language with the connectives $\neg$ ('negation'), $\wedge$ ('conjunction'), and $\vee$ ('disjunction'), which is defined over a (countable) set $\mathcal{A}$ of propositional variables or *atoms*. We use $p, q, r, \ldots$ as meta-variables for atoms. The syntax of $\mathcal{L}$ is given in a concise fashion by the following Backus-Naur-Form:

$$A ::= p \mid \neg A \mid (A \wedge A) \mid (A \vee A).$$

We use $A, B, C, \ldots$ as meta-variables for formulas. We also refer to the formulas of $\mathcal{L}$ as *non-deontic* formulas. The *deontic language* $\mathcal{L}_D$ extends $\mathcal{L}$ with formulas in which the obligation operator $O$ has been applied to non-deontic formulas, i.e. $\mathcal{L}_D$ is the smallest set $X$ such that:

1. $\mathcal{A} \subseteq X$,
2. if $A \in \mathcal{L}$, then $O(A) \in X$, and
3. if $A, B \in X$, then $\neg A, (A \wedge B), (A \vee B) \in X$.

The formulas in $\mathcal{L}_D \setminus \mathcal{L}$ we also call *deontic* formulas. Throughout the paper, the usual notational conventions about formulas apply: outermost brackets may be omitted; $\neg$ binds stronger than $\wedge$, which in turn binds stronger than $\vee$; etc. $A \to B$ is defined as $\neg A \vee B$ and $A \leftrightarrow B$ is defined as $(A \to B) \wedge (B \to A)$.

## 2 CTDLs with Limited Replacement

The (standard) *rule of replacement* (RE) says that we can infer $OB$ from $OA$ given that $A \leftrightarrow B$ is a theorem of a certain background logic. Now the problem is this: any CTDL that contains the rule of replacement and takes classical logic as its background logic either violates **No Deontic Explosion** or **Minimal Deontic Laws**. The argument is relatively straightforward. Suppose that a CTDL contains (RE) and classical logic as its background logic. Now according to **Minimal Deontic Laws**, it also contains (M). Then, using (RE) and (M), we can derive the *rule of monotonicity* (RM), which says that we can infer $OB$ from $OA$, if $A \to B$ is a theorem.[3] Since our background logic is classical, (RM) immediately results in deontic explosion: it follows from $\neg(A_1 \wedge \ldots \wedge A_n)$ being provable that $A_1 \wedge \ldots \wedge A_n \to B$ is provable; but then we can infer $OB$ from $OA_1, \ldots, OA_n$ using (RM) and (AGG), in direct contradiction to **No Deontic Explosion**. Hence no CTDL with classical logic as its background logic can satisfy both **No Deontic Explosion** and **Minimal Deontic Laws**.

In fact, (RM) is *equivalent* to (RE) and (M) in the sense that any deontic logic that has (RM) has (RE) and (M) and vice versa.[4] In other words, in the light of the rule (RE) and classical logic, we cannot distinguish between the problematic rule (RM) and the desired axiom (M). This observation motivates restricting the replacement rule, giving us the class of CTDLs with *limited replacement* [11, §5.4].[5] The idea is that we no longer sanction the inference from $OA$ to $OB$ only on the basis of $A \leftrightarrow B$ being provable, but rather demand a stronger form of equivalence to hold between $A$ and $B$ as a side-condition for replacement. The question is: What does a plausible such notion of equivalence look like?

Goble proposes an interesting CTDL with limited replacement, which instead of (classical) logical equivalence, uses the stronger condition of 'analytic equivalence' as the condition for replacement in deontic contexts.[6] Goble formalizes this notion using the binary operator $\Leftrightarrow_A$, which operates on non-deontic formulas. More specifically, an *analytic equivalence claim* is a statement of the form $A \Leftrightarrow_A B$, where $A, B \in \mathcal{L}$. Goble proposes the following axiomatization for analytic equivalences [11, p. 316]:[7]

[3]Suppose $A \to B$ is a theorem. By classical logic we get $A \leftrightarrow A \wedge B$ as a theorem. Now suppose that we have a derivation of $OA$. Using (RM), we can infer $O(A \wedge B)$. Using (M), we can derive $OB$.

[4]Above we've shown how to derive (RM) from (RE) and (M) using classical logic. For the converse direction, simply note that (RE) follows by applying (RM) "in both directions" and (M) follows from (RM) using that fact that, in classical logic, from $A \wedge B$, we can infer $A$ and $B$.

[5]Another possible response to the problem sketched above is, of course, to abandon classical logic as the background logic for our CTDL. For a discussion of this approach, see [11, p. 321–26].

[6]Restricting replacement in the context of a deontic logic is a common strategy to avoid paradoxes. Straßer and Beirlaen [15], for example, propose a very similar deontic logic based on a system they call *equivalence logic* (EL).

[7]Goble includes the axiom $(A \to B) \Leftrightarrow_A (\neg A \vee B)$, which we don't include here since we define $A \to B$ as $\neg A \vee B$.

**Axioms**:

| | |
|---|---|
| $A \Leftrightarrow_A A$ | $A \Leftrightarrow_A \neg\neg A$ |
| $A \Leftrightarrow_A (A \wedge A)$ | $A \Leftrightarrow_A (A \vee A)$ |
| $(A \wedge B) \Leftrightarrow_A (B \wedge A)$ | $(A \vee B) \Leftrightarrow_A (B \vee A)$ |
| $(A \wedge (B \wedge C)) \Leftrightarrow_A ((A \wedge B) \wedge C)$ | $(A \vee (B \vee C)) \Leftrightarrow_A ((A \vee B) \vee C)$ |
| $(A \wedge (B \vee C)) \Leftrightarrow_A ((A \wedge B) \vee (A \wedge C))$ | $(A \vee (B \wedge C)) \Leftrightarrow_A ((A \vee B) \wedge (A \vee C))$ |
| $(\neg A \wedge \neg B) \Leftrightarrow_A \neg(A \vee B)$ | $(\neg A \vee \neg B) \Leftrightarrow_A \neg(A \wedge B)$ |

**Rules**:

| | |
|---|---|
| (R1) $A \Leftrightarrow_A B / B \Leftrightarrow_A A$ | (R2) $A \Leftrightarrow_A B, B \Leftrightarrow_A C / A \Leftrightarrow_A C$ |
| (R3) $A \Leftrightarrow_A B / (A \wedge C) \Leftrightarrow_A (B \wedge C)$ | (R4) $A \Leftrightarrow_A B / (A \vee C) \Leftrightarrow_A (B \vee C)$ |
| (R5) $A \Leftrightarrow_A B / \neg A \Leftrightarrow_A \neg B$ | |

It turns out that Goble's system is deductively equivalent to the well-understood system AC of analytic containment, which is due to R.B. Angell [1, 2]. In [8], Fine gives an axiomatization of AC that is almost identical to Goble's system, except that Fine's system neither has the identity axiom $A \Leftrightarrow_A A$ nor the negation replacement rule $A \Leftrightarrow_A B / \neg A \Leftrightarrow \neg B$ (R5). It's easily shown that we can derive the identity axiom from the other axioms of Goble's system, which makes the axiom redundant:

1. $A \Leftrightarrow_A \neg\neg A$ (Axiom)
2. $\neg\neg A \Leftrightarrow_A A$ (from 1. using R1)
3. $A \Leftrightarrow_A A$ (from 1. and 2. using R2)

Moreover, Fine shows that the negation replacement rule is admissible in the system (Theorem 2, [8, p. 203]). From this it follows that Goble's system indeed is just AC.

**Proposition 2.1.** *$A \Leftrightarrow_A B$ is provable in Goble's system iff $A \Leftrightarrow_A B$ is provable in AC.*

In a sense, this observation vindicates Goble's choice of system: it turns out that Goble's system coincides with a well-known system for a non-classical notion of equivalence, which has been independently studied by philosophers and logicians [5, 6, 8]. Moreover, there is a semantics for AC, which, as we'll show in the next section, can be extended in a natural fashion to account for deontic formulas.

Goble now defines a CTDL in which replacement is restricted to analytic equivalent formulas. The system **BDL** of 'basic deontic logic' is formulated in $\mathcal{L}_D$ and consists of classical propositional logic, plus (DDS), (M), and (AGG), as well as the replacement rule (RBE), which allows us to infer $OB$ from $OA$ given that $\vdash_{AC} A \Leftrightarrow_A B$ [11, p. 314]. Note that AC plays the role of a "background system" here: derivability in AC of a certain analytic equivalence is a side-condition for the rule (RBE) in the system **BDL**, but AC itself is not part of **BDL**. This raises the question of how one can build a system in which one can explicitly reason with statements expressing analytic equivalences. This, however, goes beyond the scope of our paper.

**BDL** and $\mathbf{BDL}^-$

The system **BDL** has the following axioms and rules:

*Axioms*

1. all substitution instances of classical tautologies over the language $\mathcal{L}_D$
2. $O(A \wedge B) \to (OA \wedge OB)$ (M)
3. $(OA \wedge OB) \to O(A \wedge B)$ (AGG)
4. $(O(A \vee B) \wedge O\neg A) \to OB$ (DDS)

*Rules*

$$\frac{A \to B \quad A}{B}\ (MP) \quad \frac{OA}{OB}\ {\vdash_{AC} A \Leftrightarrow_A B}\ (RBE)$$

The system $\mathbf{BDL}^-$ is **BDL** without (DDS).

We denote derivability in $\mathbf{BDL}^-$ by $\vdash_{\mathbf{BDL}^-}$ and derivability in **BDL** by $\vdash_{\mathbf{BDL}}$. If it's clear from the context which system we're talking about, we may omit the subscript.

Before we start with semantics, let us briefly point out a few facts that'll turn out to be useful later in the paper.

First, note that the following rule

$$\frac{OA}{OB}\ {\vdash_{AC} A \wedge B \Leftrightarrow_A A}\ (RBM)$$

is derivable in both $\mathbf{BDL}^-$ and **BDL** using (RBE) and (M). We call it (RBM) by analogy with (RBE) and with (RM).

Second, note that using the rule (RBM), we can show that

$$\vdash_{\mathbf{BDL}^-} O(A \wedge B) \to O(A \vee B)$$

by observing that $\vdash_{AC} (A \wedge B) \wedge (A \vee B) \Leftrightarrow_A (A \wedge B)$.

# 3 Truthmaker Semantics for $\mathbf{BDL}^-$

Fine formulates his semantics for AC in a modified version of Bas van Fraassen's truthmaker semantics [10]. Van Fraassen originally used his semantics to give a characterization of what's effectively the 4-valued logic of First Degree Entailment, and in his paper, Fine shows how the semantics can be extended to AC [8].

Here we present a slightly different version of the semantics, which as far as we can see, was first suggested in passing by Stephen Yablo [17, p. 57]. The semantics is developed against a background theory of fine-grained *states of affairs*. These states are taken to be primitive entities of the semantics, not reducible to the possible worlds where they obtain or the like. Formally speaking, a state is a set of pairs of atoms and truth-values:

**Definition 3.1** (State)**.** *$\sigma$ is a state iff $\sigma \subseteq \mathcal{A} \times \{0, 1\}$.*[8]

States can be "negated" in a natural way:

**Definition 3.2** (Negation of a state)**.** *Let $\sigma$ be a state. We write $\overline{\sigma}$ for the negation of $\sigma$, and define $\overline{\sigma} := \{(p_i, 1 - x) : (p_i, x) \in \sigma\}$.*

Philosophically speaking, we can think of our states as *Ersatz*-states in the same way that Carnapian state descriptions are *Ersatz*-worlds, see [4]. We read the state $\{(p_1, x_1), (p_2, x_2), \ldots\}$ as the state of $p_1$ having the truth-value $x_1$, $p_2$ having the truth-value $x_2$, $\ldots$.

Note that this definition allows for incomplete and inconsistent states. A state $\sigma$ is said to be *incomplete* iff there is a $p \in \mathcal{A}$ such that neither $(p, 1) \in \sigma$ nor $(p, 0) \in \sigma$. And $\sigma$ is said to be *inconsistent* iff both $(p, 1) \in \sigma$ and $(p, 0) \in \sigma$ for some $p \in \mathcal{A}$. Intuitively, an incomplete state is one that fails to settle a certain subject matter, and an inconsistent state is one that is over-determined with respect to some subject matter.

We can think of classical valuations as special kinds of states in the following way:

**Definition 3.3** (Classical Valuation)**.** *$\omega$ is a classical valuation iff $\omega$ is a state and for every $p \in \mathcal{A}$, either $(p, 1) \in \omega$ or $(p, 0) \in \omega$ and not both.*

In fact, mathematically speaking, that's just what valuations are: functions from $\mathcal{A}$ to $\{0, 1\}$. Philosophically speaking, classical valuations are just *Ersatz*-worlds: they are technical stand-ins for ways the world can be. In what follows, we will use the familiar notion for valuations $\omega$, i.e. $\omega(p) = 1$ for $(p, 1) \in \omega$, and $\omega(p) = 0$ for $(p, 0) \in \omega$.

The idea behind truthmaker semantics is that states are those things in the world that make statements true or false. Fine gives us the following informal characterization of truthmaking: he says that $\sigma$ is a truthmaker of $A$ just in case (i) $\sigma$ necessitates the truth of $A$ and (ii) $\sigma$ is wholly relevant to the truth of $A$ [9, p. 559].[9] Note the requirement (ii) that a state be *wholly relevant* to the truth of a statement. The idea is, roughly, that in order for a state to count as a truthmaker for $A$ it may not contain any part that is irrelevant to $A$'s truth. Consider the the state of Socrates being a bald philosopher, for example. Clearly, the state necessitates the truth of the statement "Socrates is a philosopher"—necessarily, if Socrates is a bald philosopher, then "Socrates is a philosopher" is true. But, and that's the crucial point, the state is not *wholly relevant* to the

[8]Note that this includes the empty state $\emptyset$ among the states. This is, however, technically innocuous, since the empty state will not play any technical role in what follows.

[9]More precisely, this is what Fine calls "exact" truthmaking to distinguish it from truthmaking in other senses. Unless further specified, whenever we speak of the truthmakers of a statement, this is what we mean.

truth of the statement since it contains the state of Socrates being bald as a part, which is irrelevant to the truth of "Socrates is a philosopher."[10]

There is also an analogous *false*making relation, which holds between a state $\sigma$ and a statement $A$ just in case (i') $\sigma$ necessitates the falsehood of $A$ and (ii') $\sigma$ is wholly relevant to the falsehood of $A$.

In our present setting, there will be precisely one state that necessitates the truth of an atom $p$ in a wholly relevant way, which is just the state $\{(p, 1)\}$. And there is exactly one state that necessitates the falsehood of $p$ in a wholly relevant way, which is $\{(p, 0)\}$. If we start from this and take Fine's recursive clauses for the truthmaking and falsemaking relation, we end up with the following definition (cf. [8, p. 205–6][11]):

**Definition 3.4** (Truthmakers, Falsemakers)**.** *For all $A \in \mathcal{L}$, the set $[A]^+$ of truthmakers of $A$ and the set $[A]^-$ of falsemakers of $A$ is defined by simultaneous recursion as follows:*

*i) a)* $[p]^+ = \{\{(p, 1)\}\}$
   *b)* $[p]^- = \{\{(p, 0)\}\}$

*ii) a)* $[\neg A]^+ = [A]^-$
   *b)* $[\neg A]^- = [A]^+$

*iii) a)* $[A \wedge B]^+ = \{\sigma \cup \tau : \sigma \in [A]^+, \tau \in [B]^+\}$
   *b)* $[A \wedge B]^- = [A]^- \cup [B]^- \cup [A \vee B]^-$

*iv) a)* $[A \vee B]^+ = [A]^+ \cup [B]^+ \cup [A \wedge B]^+$
   *b)* $[A \vee B]^- = \{\sigma \cup \tau : \sigma \in [A]^-, \tau \in [B]^-\}$

One way of thinking about what's going on here is that we're keeping track of the precise truth and falsity conditions of a given statement: the members of $[A]^+$ are the exact conditions that need to be satisfied by a valuation for $A$ to be true under the valuation, and the members of $[A]^-$ are the exact conditions for $A$'s falsehood.

Fine [8] discusses various notions of semantic content that can be developed in the truthmaker setting. To get a semantics for AC, we need the notion of *replete* content, which is defined in terms of convexity:

**Definition 3.5** (Convex set)**.** *A set $\Sigma$ of states is said to be* convex *if and only if for all states $\sigma, \tau, \delta$, if $\sigma, \delta \in \Sigma$ and $\sigma \subseteq \tau \subseteq \delta$, then $\tau \in \Sigma$.*

Every set of states can canonically be transformed into a convex set, by filling in the missing pieces:

---

[10] For further explanation, see [9, p. 559].

[11] Here, we work with what Fine calls the "inclusive" clauses for truthmaking. Fine also discusses a non-inclusive version of the semantics, which we don't need in the present paper. Moreover, we work with a structure that is (model-isomorphic to) the canonical model Fine uses in his paper. From a logical point of view, as we'll explain below, restricting ourselves to the canonical model doesn't change anything with respect the soundness and completeness of AC in our semantics. We chose the current presentation of the semantics for ease of exposition.

**Definition 3.6** (Convex closure)**.** *We define the convex closure, $conv(\Sigma)$, of a set of states $\Sigma$ as the smallest $\Theta$ such that $\Sigma \subseteq \Theta$ and $\Theta$ is convex, i.e.*

$$conv(\Sigma) = \bigcap \{\Theta : \Sigma \subseteq \Theta, \Theta \textit{ is convex}\}.$$

*For $conv([A]^+)$ we also write $[\![A]\!]^+$, and for $conv([A]^-)$ we write $[\![A]\!]^-$.*

It follows from basic set-theory that $conv(\Sigma)$ is well-defined. It is worthwhile, however, to note the following facts about the operation:

**Lemma 3.7** (Properties of convex closure)**.** *We have:*

1. *$conv(\Sigma)$ is convex*
2. *$\Sigma \subseteq conv(\Sigma)$*
3. *If $\Sigma \subseteq \Delta$, then $conv(\Sigma) \subseteq conv(\Delta)$*
4. *$conv(\Sigma) = \Sigma \cup \{\sigma : \exists \tau, \pi \in \Sigma : \tau \subseteq \sigma \subseteq \pi\}$*

*Proof.*

1. Suppose, for proof by contradiction, that $conv(\Sigma)$ is not convex, i.e. there are $\sigma, \tau \in conv(\Sigma)$ and a $\pi$ with $\sigma \subseteq \pi \subseteq \tau$ but $\pi \notin conv(\Sigma)$. Since $\sigma, \tau \in conv(\Sigma)$ and $conv(\Sigma) = \bigcap\{\Theta : \Sigma \subseteq \Theta, \Theta \text{ is convex}\}$, we know that for all $\Theta$ such that $\Sigma \subseteq \Theta$ and $\Theta$ convex, $\sigma, \tau \in \Theta$. Pick an arbitrary such $\Theta$. Since $\sigma \subseteq \pi \subseteq \tau$ and $\Theta$ is convex, we know that $\pi \in \Theta$. But then $\pi \in \Theta$ *for all* $\Sigma \subseteq \Theta$ and $\Theta$ is convex. But then $\pi \in \bigcap\{\Theta : \Sigma \subseteq \Theta, \Theta \text{ is convex}\} = conv(\Sigma)$. Contradiction.

2. Obviously, $\Sigma \subseteq \Theta$, for all $\Theta$ such that $\Sigma \subseteq \Theta$ and $\Theta$ is convex. It follows immediately that $\Sigma \subseteq \bigcap\{\Theta : \Sigma \subseteq \Theta, \Theta \text{ is convex}\}$.

3. Note that, since $\Sigma \subseteq \Delta$, $\Sigma \subseteq \Theta$ for all $\Theta$ such that $\Delta \subseteq \Theta$ and $\Theta$ is convex. But from this it follows that $\{\Theta : \Delta \subseteq \Theta, \Theta \text{ convex}\} \subseteq \{\Theta : \Sigma \subseteq \Theta, \Theta \text{ convex}\}$. But then, by basic set-theory,

$$\underbrace{\bigcap\{\Theta : \Sigma \subseteq \Theta, \Theta \text{ convex}\}}_{=conv(\Sigma)} \subseteq \underbrace{\bigcap\{\Theta : \Delta \subseteq \Theta, \Theta \text{ convex}\}}_{=conv(\Delta)}.$$

4. $conv(\Sigma) \subseteq \Sigma \cup \{\sigma : \exists \tau, \pi \in \Sigma : \tau \subseteq \sigma \subseteq \pi\}$: To see this, note that $\Sigma \subseteq \Theta$ and $\{\sigma : \exists \tau, \pi \in \Sigma : \tau \subseteq \sigma \subseteq \pi\} \subseteq \Theta$, for all $\Theta$ such that $\Sigma \subseteq \Theta$ and $\Theta$ is convex. Hence $\bigcap\{\Theta : \Sigma \subseteq \Theta, \Theta \text{ is convex}\} = conv(\Sigma) \subseteq \Sigma \cup \{\sigma : \exists \tau, \pi \in \Sigma : \tau \subseteq \sigma \subseteq \pi\}$, as desired.

   $\Sigma \cup \{\sigma : \exists \tau, \pi \in \Sigma : \tau \subseteq \sigma \subseteq \pi\} \subseteq conv(\Sigma)$: Suppose that $\sigma \in \Sigma \cup \{\sigma : \exists \tau, \pi \in \Sigma : \tau \subseteq \sigma \subseteq \pi\}$. We can infer that either (a) $\sigma \in \Sigma$ or (b) $\sigma$ is such there are $\tau, \pi \in \Sigma$ with $\tau \subseteq \sigma \subseteq \pi$. If (a), then, by 2., $\sigma \in conv(\Sigma)$. If (b), then we know that $\tau, \pi \in conv(\Sigma)$ since $\tau, \pi \in \Sigma$ and $\Sigma \subseteq conv(\Sigma)$ by 2. But by 1. $conv(\Sigma)$ is convex and so $\sigma \in conv(\Sigma)$.

□

Following Fine, we call $[\![A]\!]^+$ the set of *replete truthmakers* of $A$ and $[\![A]\!]^-$ *replete falsemakers* of $A$. To illustrate the difference between exact (in the sense of inclusive semantics truthmaker semantics as developed in definition 3.4) and replete truthmakers, it's maybe helpful to look at an example. Take the formula $(p \wedge q \wedge r) \vee p$. This formula has two exact truthmakers: the state that makes the left disjunct (i.e. $(p \wedge q \wedge r)$) true, and the state that makes the right disjunct (i.e. $p$) true, and this is why we have $[(p \wedge q \wedge r) \vee p]^+ = \{\{(p,1),(q,1),(r,1)\},\{(p,1)\}\}$. The set of replete truthmakers for the same formula now contains additional elements, in our example $[\![(p \wedge q \wedge r) \vee p]\!]^+ = \{\{(p,1),(q,1),(r,1)\},\{(p,1),(q,1)\},\{(p,1),(r,1)\},\{(p,1)\}\}$. Intuitively speaking, the set of replete truthmakers (falsemakers) also contains all those states that make the formula true (false) but that are also contained in the statement's subject matter.

Finally, note that under the present semantics, truth and falsemakers are closed under union:

**Lemma 3.8.** *For all $A \in \mathcal{L}$, if $\sigma, \tau \in [A]^+$, then $\sigma \cup \tau \in [A]^+$ and if $\sigma, \tau \in [A]^-$, then $\sigma \cup \tau \in [A]^-$.*

*Proof.* By induction on complexity. □

Intuitively, this means that the "fusion" of two truthmakers (falsemakers) still is a truthmaker (falsemaker).[12] This fact carries over to replete truthmakers:

**Corollary 3.9.** *For all $A \in \mathcal{L}$, if $\sigma, \tau \in [\![A]\!]^+$, then $\sigma \cup \tau \in [\![A]\!]^+$ and if $\sigma, \tau \in [\![A]\!]^-$, then $\sigma \cup \tau \in [\![A]\!]^-$.*

It can be shown that Goble's axioms and rules for analytic equivalence exactly describe identity of replete truthmakers (or falsemakers, for that matter):

**Theorem 3.10** (Replete Truthmakers and Analytic Equivalence)**.** *For all $A, B \in \mathcal{L}$, the following three statements are equivalent:*

- $\vdash_{AC} A \Leftrightarrow_A B$
- $[\![A]\!]^+ = [\![B]\!]^+$
- $[\![A]\!]^- = [\![B]\!]^-$

*Proof.* This is a corollary of Fine's soundness and completeness result for his semantics for AC [8, Theorems 14 and 21]. Fine shows that $\vdash_{AC} A \Leftrightarrow_A B$ iff the replete truth and falsemakers of $A$ and $B$ are the same in a wider class of truthmaker models, to which our concrete truthmaker model structure belongs. In fact, the truthmaker model we're working with is (model-isomorphic to) Fine's *canonical* model (see footnote 11), meaning that in it, all and only those formulas have the same replete truth and falsemakers that are provable in AC. From this observation, the above result follows immediately.

□

[12] For a discussion of this, see [8, p. 206].

In this sense, replete truthmakers are the semantic foundation for analytic equivalence. Besides being an interesting observation in itself, this observation is a partial solution to Goble's challenge to find an intuitive semantical framework for **BDL**, because it gives us a semantic framework to interpret the Goble's notion of analytic equivalence.

To interpret obligations we add one final ingredient to our base model: a set $Ok$, which contains all those states that are normatively admissible. So this leads us to the following definition of a base model:

**Definition 3.11** (Base Model)**.** *A model is a tuple $\mathcal{M} = (\omega, Ok)$, where $\omega$ is a classical valuation over $\mathcal{L}$ and $Ok \subseteq \wp(\mathcal{A} \times \{0,1\})$.*

To interpret obligations, we use the following idea: what it means for $A$ to be obligatory is that *no replete falsemaker of* $A$ is admissible. Because we have $[\![A]\!]^- = [\![\neg A]\!]^+$,[13] $A$ is obligatory iff *no replete truthmaker of* $\neg A$ is admissible. Putting all these conditions together results in the following truth-conditions for all formulas of the deontic language $\mathcal{L}_D$:

**Definition 3.12** (Truth)**.** *Let $\mathbf{M} = (\omega, Ok)$ be a model. Note that the truth-conditions for $p \in \mathcal{A}$ and for non-deontic formulas are classical:*

*(i)* $\mathcal{M} \vDash p$ *iff* $\omega(p) = 1$

*(ii)* $\mathcal{M} \vDash \neg A$ *iff* $\mathcal{M} \nvDash A$

*(iii)* $\mathcal{M} \vDash A \wedge B$ *iff* $\mathcal{M} \vDash A$ *and* $\mathcal{M} \vDash B$

*(iv)* $\mathcal{M} \vDash A \vee B$ *iff* $\mathcal{M} \vDash A$ *or* $\mathcal{M} \vDash B$

*(v)* $\mathcal{M} \vDash OA$ *iff* $[\![A]\!]^- \cap Ok = \emptyset$

So $Ok$ is a set of states, i.e. a set of sets of ordered pairs of atoms and truth-values. To give you an idea of how we interpret these $Ok$ sets, consider the following example. Assume that $Ok = \{\{(p,1),(q,0)\},\{(p,1),(t,1)\}\}$. This $Ok$ set renders two states admissible: the state in which $p$ is true and $q$ is false (e.g. you drink and you don't drive), and the state in which $p$ is true and $t$ is true (e.g. you drink and you take a taxi).

So far there are no conditions on the set $Ok$ of admissible states. A consequence of that is that a complex state $\{(p,1),(t,1)\}$ can be in $Ok$, without any of its substates, e.g. $\{(p,1)\}$, being in $Ok$. This enables us to express that two things might be admissible only in combination with one another, while not being admissible in isolation.[14]

To get a semantics for $\mathbf{BDL}^-$ and **BDL**, however, we need $Ok$ to satisfy additional conditions. To validate the axioms of $\mathbf{BDL}^-$, we need to assume that if two states are inadmissible, then any state "in between" will also be

[13] To see this, note that $[A]^- = [\neg A]^+$, i.e. we have both $[A]^- \subseteq [\neg A]^+$ and $[\neg A]^+ \subseteq [A]^-$. Since $[\![A]\!]^- = conv([A]^-)$ and $[\![\neg A]\!]^+ = conv([\neg A]^+)$ by definition, Lemma 3.7.3 gives us both $[\![A[\![^- \subseteq [\![\neg A[\![^+$ and $[\![\neg A[\![^+ \subseteq [\![A[\![^-$, i.e. $[\![A]\!]^- = [\![\neg A]\!]^+$.

[14] We will come back to this closure condition for $Ok$ later in section 4 when we talk about (DDS) and a semantics for the stronger logic **BDL**.

inadmissible, and we need to assume that if the combination of two states is admissible, then at least one of the two states is admissible:

**Definition 3.13** (Reverse Convexity)**.** *A set of states* $\Sigma$ *is* reverse convex *iff the complement of* $\Sigma$ *with respect to the set of all states,* $\Sigma^C$*, is convex, i.e. for all* $\sigma, \tau, \delta$*: if* $\sigma \notin \Sigma$ *and* $\tau \notin \Sigma$ *and* $\sigma \subseteq \delta \subseteq \tau$*, then* $\delta \notin \Sigma$*.*

**Definition 3.14** (Reverse Closure)**.** *A set of states* $\Sigma$ *is* reverse closed *iff* $\Sigma^C$ *is closed under union, i.e. for any two states* $\sigma, \tau \notin \Sigma$*, we have* $\sigma \cup \tau \notin \Sigma$*.*

To illustrate what the reverse convexity condition says, consider an example in which it is violated. For instance, take the $Ok$ set that only renders the state admissible which makes $p$ and $r$ true, $Ok = \{\{(p, 1), (r, 1)\}\}$. As a consequence, we have $\{(p, 1), (r, 1), (s, 1)\} \notin Ok$ and $\{(p, 1)\} \notin Ok$. So neither is the state admissible which makes $p$, $r$ and $s$ true, nor is the state admissible which makes $p$ true. Intuitively speaking, this means that there is an admissible state such that neither a stronger nor a weaker state is admissible. Reverse convexity excludes this situation, i.e. it excludes $Ok$ sets like the one in this example.[15]

These two conditions now finally give us the notion of a $\mathbf{BDL}^-$ model:

**Definition 3.15** ($\mathbf{BDL}^-$ model)**.** *A* $\mathbf{BDL}^-$ *model is model* $\mathcal{M} = (\omega, Ok)$ *such that* $Ok$ *is reverse convex and reverse closed.*[16]

Validity and logical consequence are defined as usual:

**Definition 3.16** (Validity)**.** *For all* $A \in \mathcal{L}_D$*,*

$$\vDash A \text{ iff for all } \mathbf{BDL}^- \text{ models } \mathcal{M}\text{: } \mathcal{M} \vDash A.$$

**Definition 3.17** (Logical Consequence)**.** *For all* $\Phi \subseteq \mathcal{L}_D$ *and* $A \in \mathcal{L}_D$*,*

$$\Phi \vDash A \text{ iff for all } \mathbf{BDL}^- \text{ models } \mathcal{M}, \text{ if } \mathcal{M} \vDash \Phi, \text{ then } \mathcal{M} \vDash A.$$

To illustrate how our semantics works, we now consider two concrete models. The first one shows that, although the semantics validates (M), obligations are not generally closed under logical consequence. In particular, we show that that weakening in the form of $OA \to O(A \vee B)$ is not valid. The second one shows that normative conflicts are satisfiable.

(i) Let $\mathcal{M} = (\omega, Ok)$ with $Ok = \{\sigma : (q, 0) \in \sigma\}$. It's easily checked, then, that $Ok$ so defined is both reverse convex and reverse closed. And since $[\![p]\!]^- = \{\{(p, 0)\}\}$, we have $[\![p]\!]^- \cap Ok = \emptyset$ and so by the truth-condition for obligation formulas $\mathcal{M} \vDash Op$. However, in light of the fact that $[\![p \vee q]\!]^- = \{\{(p, 0), (q, 0)\}\}$ we get $[\![p \vee q]\!]^- \cap Ok \neq \emptyset$, i.e. $\mathcal{M} \nvDash O(p \vee q)$. Hence, we get $\mathcal{M} \nvDash Op \to O(p \vee q)$.

[15] Note that both conditions (reverse convexity and reverse closure) really concern the complement of a set of states. Why this is so, will become apparent soon.

[16] Note that in many cases, the condition of reverse convexity forces us to include $\emptyset$ among the members of $Ok$. Otherwise, we'd get trivial counterexamples. Consider, e.g., any set $Ok$ such that for all $\sigma, \tau \in Ok$, $\sigma \cap \tau = \emptyset$. If then $\emptyset \notin Ok$, the set would not be reverse convex, since $\emptyset \subseteq \sigma$ for *all* $\sigma$. Intuitively, however, this is not a serious constraint, since $\emptyset$ corresponds to the empty state, which obtains no matter what. Including this state among the admissible states is philosophically innocuous.

(ii) Let $\mathcal{M} = (\omega, Ok)$ with $Ok = \{\emptyset\} \cup \{\{(p_i, 1)\} : i \geq 2 \text{ and } p_i \in \mathcal{A}\}$.[17] Clearly, $Ok$ so defined is reverse closed and reverse convex. $Ok$ renders a state admissible that makes $p_2$ true, one that makes $p_3$ true, and so on. According to the truth-condition for obligations, this makes $\neg p_2$, $\neg p_3$, etc. not obligatory. So we have: $\mathcal{M} \vDash \neg O\neg p_i$, for all $i \geq 2$. What about $p_1$, though? Since for all $p \in \mathcal{A}$ we have $[p]^+ = [\![p]\!]^+$ and $[p]^- = [\![p]\!]^-$, there is neither a truthmaker nor a falsemaker of $p_1$ in $Ok$, i.e. $[\![p_1]\!]^- \cap Ok = \emptyset$ and $[\![\neg p_1]\!]^- \cap Ok = \emptyset$. And this means that $\mathcal{M} \vDash Op_1$ and $\mathcal{M} \vDash O\neg p_1$, i.e. $\mathcal{M}$ satisfies a normative conflict. Despite the fact that normative conflicts are satisfiable, it is also easily shown that $\mathbf{BDL}^-$'s base logic is classical:

**Lemma 3.18** (Classicality). *For all $A \in \mathcal{L}_D$ and all $\mathbf{BDL}^-$ models, either $\mathcal{M} \vDash A$ or $\mathcal{M} \nvDash A$ and never both.*

# 4 On Deontic Disjunctive Syllogism and BDL

In the semantics for $\mathbf{BDL}^-$, (DDS) is not valid. It's easy to construct a countermodel: let $\mathcal{M} = (\omega, Ok)$ be such that

- $\omega$ is arbitrary, and
- $Ok = \{\emptyset, \{(q, 0)\}\}$.

Note that $Ok$ is reverse convex and reverse closed. We have $\mathcal{M} \vDash O\neg p$, $\mathcal{M} \vDash O(p \vee q)$ but $\mathcal{M} \nvDash Oq$, since $[\![q]\!]^- = \{\{(q, 0)\}\} \cap Ok \neq \emptyset$. Hence (DDS) is not a valid schema for our semantics.

To obtain a sound and complete semantics for **BDL** we need a condition on $\mathbf{BDL}^-$ models which guarantees that (DDS) preserves truth in the models that satisfy the condition. This is what we set out to do in this section.

First, we need some further auxiliary concepts. Just like we can negate states, we can also negate finite sets of states, i.e. *contents*. Note that infinite sets of states don't play the role of semantic values of formulas in our language, and thus the restriction to finite states and finite sets of states in the following is semantically innocuous.[18]

**Definition 4.1** (Negation of a content). *Let $\sigma_1, \ldots, \sigma_n$ be states. We define:*

$$\overline{\{\sigma_1, \ldots, \sigma_n\}} = \{\overline{\{f(\sigma_1), \ldots, f(\sigma_n)\}} : f : \{\sigma_1, \ldots, \sigma_n\} \rightarrow \bigcup_{i=1}^{n} \sigma_i \textit{ is a choice function}\}$$ [19]

For simplicity, let's also introduce the notion of *fusions* of contents:

**Definition 4.2** (Fusion of Contents). *We call $\Sigma \circ \Lambda$ the fusion of two sets of sets $\Sigma$ and $\Lambda$, and define $\Sigma \circ \Lambda = \{\sigma \cup \tau : \sigma \in \Sigma \text{ and } \tau \in \Lambda\}$.*

[17] Note that we need to include $\emptyset$ in $Ok$ to ensure that it is reverse convex. See previous footnote.

[18] This claim can be established by a simple induction on the complexity of formulas.

[19] Here a *choice function* is understood in the standard set-theoretic sense as a function $f$ on sets such that for all sets $A$ in the domain of the function, $f(A) \in A$.

We'll use the following lemma every now and then later in the paper:

**Lemma 4.3.** *For all $A \in \mathcal{L}$, we have $[A]^+ \circ [A]^+ = [A]^+$ and $[A]^- \circ [A]^- = [A]^-$.*

*Proof.* This follows from the fact that truth and falsemakers are closed under fusions (Lemma 3.8). □

**Lemma 4.4.** *For all $A \in \mathcal{L}$,*

- $\overline{[A]^+} = [\neg A]^+$
- $\overline{[A]^-} = [\neg A]^-$

*Proof.* By induction on complexity. For the base case, note that:

- $[p]^+ = [\neg p]^- = \{\{(p,1)\}\}$ and $\overline{\{\{(p,1)\}\}} = \{\{(p,0)\}\}$
- $[\neg p]^+ = [p]^- = \{\{(p,0)\}\}$ and $\overline{\{\{(p,0)\}\}} = \{\{(p,1)\}\}$

For $\neg A$, note that

- $\overline{[\neg A]^+} = \overline{[A]^-} \overset{IH}{=} [\neg A]^- = [A]^+ = [\neg\neg A]^+$
- $\overline{[\neg A]^-} = \overline{[A]^+} \overset{IH}{=} [\neg A]^- = [A]^- = [\neg\neg A]^-$

We only consider the case for $A \vee B$, leaving the case for $A \wedge B$ to the interested reader. We first note the following identity:

$$\overline{\Sigma \circ \Delta} = \overline{\Sigma} \cup \overline{\Delta}. \tag{$*$}$$

Applying this repeatedly, we get the following argument:

- $\overline{[A \vee B]^+} = \overline{[A]^+ \cup [B]^+ \cup ([A]^+ \circ [B]^+)} \overset{*}{=} \overline{[A]^+} \circ \overline{[B]^+} \circ \overline{[A]^+ \cup [B]^+} \overset{*}{=} \overline{[A]^+} \circ \overline{[B]^+} \circ \overline{[A]^+} \circ \overline{[B]^+} = \overline{[A]^+} \circ \overline{[B]^+} \overset{IH}{=} [\neg A]^+ \circ [\neg B]^+ = \ldots = [\neg(A \vee B)]^+$
- $\overline{[A \vee B]^-} = \overline{[A]^- \circ [B]^-} \underset{\text{Lemma 4.3}}{=} \overline{[A]^- \circ [B]^- \circ ([A]^- \cup [B]^-)} \overset{*}{=} \overline{[A]^-} \cup \overline{[B]^-} \cup \overline{[A]^-} \circ \overline{[B]^-} \overset{IH}{=} [\neg A]^- \cup [\neg B]^- \cup [\neg A]^- \circ [\neg B]^- = \ldots = [\neg(A \vee B)]^-$

□

It immediately follows:

**Corollary 4.5.**

1. $[\![\neg A]\!]^+ = conv(\overline{[A]^+})$
2. $[\![\neg A]\!]^- = conv(\overline{[A]^-})$

**Definition 4.6** (Trace formulas)**.** *For all finite states $\sigma$, we set*

$$tr(\sigma) = \bigwedge(\{p : (p,1) \in \sigma\} \cup \{\neg p : (p,0) \in \sigma\})$$

*And for all finite sets of finite states $\Sigma$, we set:*

$$tr(\Sigma) = \bigvee\{tr(\sigma) : \sigma \in \Sigma.\}$$

Note that we only define trace formulas for finite states and finite sets of finite states. Again, this is because infinite states and infinite sets of states don't play a role in the present semantics.

**Lemma 4.7.** *For all finite states $\sigma$ and all finite sets of finite states $\Sigma$,*

1. $[tr(\sigma)]^+ = \{\sigma\}$
2. $[tr(\Sigma)]^+ = \Sigma$

The following theorem illustrates that (DDS)'s validity corresponds to a certain closure property of $Ok$ sets:

**Theorem 4.8.** *Let $\mathfrak{M}$ be a class of* $\mathbf{BDL}^-$*-models. Then, the following are equivalent:*

1. *for all $A, B \in \mathcal{L}$ and for all $\mathcal{M} \in \mathfrak{M}$, $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$,*
2. *for all $\mathcal{M} = (\omega, Ok) \in \mathfrak{M}$, for all finite sets of finite states $\Sigma, \Delta$, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$, then $conv(\overline{\Delta}) \cap Ok = \emptyset$.*

*Proof.* 1. $\Rightarrow$ 2. Assume that for all $A, B, C \in \mathcal{L}$ and for all $\mathcal{M} \in \mathfrak{M}$, $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$. For proof by contradiction, assume that there is an $\mathcal{M}$ and $\Sigma, \Delta$ with $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$ but $conv(\overline{\Delta}) \cap Ok \neq \emptyset$. Consider $O(tr(\Sigma) \vee tr(\Delta))$, $O\neg tr(\Sigma)$, and $Otr(\Delta)$. Using Lemmas 4.4 and 4.7, we get $[tr(\Sigma) \vee tr(\Delta)]^- = [tr(\Sigma)]^- \circ [tr(\Delta)]^- = \overline{\Sigma} \circ \overline{\Delta}$. So $[\![tr(\Sigma) \vee tr(\Delta)]\!]^+ = conv(\overline{\Sigma} \circ \overline{\Delta})$. But then, we get $\mathcal{M} \vDash O(tr(\Sigma) \vee tr(\Delta))$, since by assumption $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$. In a similar fashion, we can determine that $\mathcal{M} \vDash O\neg tr(\Sigma)$ and $\mathcal{M} \nvDash Otr(\Delta)$, which is in contradiction to the assumption that $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$, for all $A, B$. Hence for all $\mathcal{M} = (\omega, Ok) \in \mathfrak{M}$, for all sets of states $\Sigma, \Delta$, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\overline{\Sigma}) \cap Ok = \emptyset$, then $conv(\Delta) \cap Ok = \emptyset$.

2. $\Rightarrow$ 1. Conversely, assume that for all $\mathcal{M} = (\omega, Ok) \in \mathfrak{M}$, for all sets of states $\Sigma, \Delta$, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$, then $conv(\overline{\Delta}) \cap Ok = \emptyset$. This means that for any formulas $A, B$, if

$$\underbrace{conv(\overline{[A]^+} \circ \overline{[B]^+})}_{=[\![A \vee B]\!]^-} \cap Ok = \emptyset$$

and

$$\underbrace{conv([A]^+)}_{=[\![\neg A]\!]^-} \cap Ok = \emptyset$$

then

$$\underbrace{conv(\overline{[B]^+})}_{=[\![B]\!]^-} \cap Ok \neq \emptyset$$

Which is just to say that if $\mathcal{M} \vDash O(A \vee B)$ and $\mathcal{M} \vDash O\neg A$, then $\mathcal{M} \vDash OB$. In other words, $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$.

$\square$

Because of this fact we can define **BDL** models as follows:

**Definition 4.9** (**BDL** model)**.** *A **BDL** model is a $\textbf{BDL}^{-}$ model $\mathcal{M} = (\omega, Ok)$ in which $Ok$ also satisfies:*

> *For all finite sets of finite states $\Sigma, \Delta$, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$, then $conv(\overline{\Delta}) \cap Ok = \emptyset$.*

Validity and logical consequence for **BDL** are defined as usual:

**Definition 4.10** (Validity, **BDL**)**.** *For all $A \in \mathcal{L}_D$,*

$$\vDash_{\textbf{BDL}} A \text{ iff for all } \textbf{BDL} \text{ models } \mathcal{M}\text{: } \mathcal{M} \vDash A.$$

**Definition 4.11** (Logical Consequence, **BDL**)**.** *For all $\Phi \subseteq \mathcal{L}_D$ and $A \in \mathcal{L}_D$,*

$$\Phi \vDash_{\textbf{BDL}} A \text{ iff for all } \textbf{BDL} \text{ models } \mathcal{M}, \text{ if } \mathcal{M} \vDash \Phi, \text{ then } \mathcal{M} \vDash A.$$

This finally gives us a semantics for full **BDL**.

We now conclude this section with two observation concerning (DDS).

The first observation shows that (DDS) is incompatible with a property we might want the set $Ok$ of admissible states to satisfy.

As we have seen in one of the examples at the end of section 3, the $Ok$ set is not closed under subsets. So an $Ok$ set of a **BDL** model does not have to satisfy the following closure property:

**Definition 4.12** (Closure under subsets)**.** *A set of states $\Sigma$ is* closed under subsets *iff for all $\sigma \in \Sigma$, we also have $\sigma' \in \Sigma$, whenever $\sigma' \subseteq \sigma$.*

That $Ok$ sets do not have to satisfy closure under subset means that a state can be admissible without all of its parts being admissible. But this seems odd: how can a complex state be admissible and still contain a part that is not admissible? So it seems only natural to further restrict $Ok$ such that situations like these are excluded, i.e. to require $Ok$ sets to be closed under subsets. As natural as this might look at first sight, it has disastrous consequences if combined with (DDS): as soon as we require $Ok$ sets to be closed under subsets, we end up with an extension of **BDL** that validates deontic explosion in situations of normative conflicts. To see this note that closing $Ok$ under subsets results in the validity of

**(Add)** $OA \to O(A \vee B)$.[20]

(Add) doesn't (always) sit well with (DDS) and normative conflicts.[21] The following argument is well-known, and it is as simple as it is effective: suppose that there is a normative conflict, i.e. $OA$ and $O\neg A$. It's obvious that $OA$, (Add) and (MP) result in $O(A \vee B)$, which together with $O\neg A$ and (DDS) results in $OB$, for arbitrary $B$. Hence, the logic contains a version of deontic explosion. And now the dilemma is apparent: we cannot have (DDS) and make $Ok$ sets

[20] See also the first example at the end of section 3.

[21] Goble also makes this observation in [11].

closed under subsets. Whether this speaks against (DDS) or against (Add), we leave to the reader to decide.[22]

Now to our second observation concerning (DDS). Although we show later in section 6.2 that **BDL** does not lead to (deontic) explosion in the presence of a normative conflict, **BDL** contains weaker forms of "explosion like" principles. As one of the reviewers pointed out, **BDL** contains the following principle:

**(ELP)** $OA, O\neg A, O(B \vee C) \models_{\mathbf{BDL}} OB$

*Proof.* Note that we have

**(P1)** $OA, O(B \vee C) \models_{\mathbf{BDL}^-} O(A \vee B)$.

To see this, note that from $OA$ and $O(B \vee C)$, we can infer in **BDL** that $O(A \wedge (B \vee C))$. Now, it is tedious but possible to show that $(A \wedge (B \vee C)) \wedge (A \vee B) \Leftrightarrow_A A \wedge (B \vee C)$ (in fact, this is the same situation as in the proof of Lemma 5.11). This means that from $O(A \wedge (B \vee C))$ we can infer $O(A \vee B)$ by a single application of (RBM). In other words, $OA, O(B \vee C) \vdash_{\mathbf{BDL}} O(A \vee B)$. By soundness (see Theorem 5.2 below), the claim follows. To prove (ELP), let us suppose $OA, O\neg A, O(B \vee C)$. By (P1) we get $O(A \vee B)$, but since we also have $O\neg A$, (DDS) leads to $OB$. And this concludes the proof of (ELP). □

Although (ELP) might already look quite bad by itself, in presence of a normative conflict, it renders conjunction and disjunction semantically indistinguishable within the scope of the obligation operator:

**(ConDis)** $OA, O\neg A \models_{\mathbf{BDL}} O(B \vee C) \leftrightarrow O(B \wedge C)$

*Proof.* To see this, note that (ELP) also gives us $OA, O\neg A, O(B \vee C) \models_{\mathbf{BDL}} OC$, which together with (AGG) results in $OA, O\neg A, O(B \vee C) \models_{\mathbf{BDL}} O(B \wedge C)$. So, we have that $OA, O\neg A \models_{\mathbf{BDL}} O(B \vee C) \rightarrow O(B \wedge C)$. Since, we've already observed that $O(A \wedge B) \rightarrow O(A \vee B)$ is valid in $\mathbf{BDL}^-$ and hence in **BDL**, we get the desired result. □

# 5 Main Results: Soundness & Completeness of $\mathbf{BDL}^-$ and BDL

In this section, we set out to prove the soundness and completeness of $\mathbf{BDL}^-$ with respect to our semantics. The soundness and completeness of the semantics for **BDL** then follows as a corollary from the soundness and completeness result and Theorem 4.8.

We start with soundness and observe the following lemma:

**Lemma 5.1.** *The following are equivalent for all* $\mathbf{BDL}^-$ *models* $\mathcal{M} = (\omega, Ok)$*:*

1. $[\![A \wedge B]\!]^- \cap Ok = \emptyset$

[22] Another interesting approach is to restrict DDS to premises without normative conflicts. Goble himself has been working with this weakened version of DDS in [12], Christian Straßer explored this idea further in [16].

2. $[\![A]\!]^- \cap Ok = \emptyset$ *and* $[\![B]\!]^- \cap Ok = \emptyset$

*Proof.* 1. ⇒ 2. First, note that taking the convex closure of a set of states is a monotonic operation (Lemma 3.7), i.e. for all sets of states $\Sigma$ and $\Lambda$, if $\Sigma \subseteq \Lambda$, then $conv(\Sigma) \subseteq conv(\Lambda)$. Now, remember that

$$[\![A \wedge B]\!]^- = conv(\underbrace{[A \wedge B]^-}_{=[A]^- \cup [B]^- \cup [A \vee B]^-}).$$

Since $[\![A]\!]^- = conv([A]^-)$ and $[\![B]\!]^- = conv([B]^-)$, using the aforementioned monotonicity, we get that $[\![A]\!]^-, [\![B]\!]^- \subseteq [\![A \wedge B]\!]^-$. So, if $[\![A \wedge B]\!]^- \cap Ok = \emptyset$, then also both $[\![A]\!]^- \cap Ok = \emptyset$ and $[\![B]\!]^- \cap Ok = \emptyset$.

2. ⇒ 1. To prove this direction, we need to use both conditions for $\mathbf{BDL}^-$ models, reverse convexity and reverse closure. Suppose that $[\![A]\!]^- \cap Ok = \emptyset$ and $[\![B]\!]^- \cap Ok = \emptyset$. And suppose further that $\sigma \in [\![A \wedge B]\!]^-$ for an arbitrary $\sigma$. We now show that $\sigma \notin Ok$. Since

$$[\![A \wedge B]\!]^- \underset{Lemma\ 3.7}{=} \underbrace{[A]^- \cup [B]^- \cup [A \vee B]^-}_{=[A \vee B]^-} \cup \{\sigma : \exists \tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-, \tau \subseteq \sigma \subseteq \pi\},$$

we can distinguish four cases: (i) $\sigma \in [A]^-$, (ii) $\sigma \in [B]^-$, (iii) $\sigma \in [A \vee B]^-$, and (iv) $\sigma \in \{\sigma : \exists \tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-, \tau \subseteq \sigma \subseteq \pi\}$. In case (i), since $\sigma \in [A]^-$ and $[A]^- \subseteq [\![A]\!]^-$, we get that $\sigma \in [\![A]\!]^-$. But since, by assumption, $[\![A]\!]^- \cap Ok = \emptyset$, it follows that $\sigma \notin Ok$. Case (ii) works analogously. For case (iii), assume that $\sigma \in [A \vee B]^-$, i.e. there are $\tau$ and $\pi$ such that $\sigma = \tau \cup \pi$ with $\tau \in [A]^-$ and $\pi \in [B]^-$. If there are such $\tau$ and $\pi$, we know that $\tau, \pi \notin Ok$ (by the arguments from cases (i) and (ii)). So we can conclude that $\tau \cup \pi = \sigma \notin Ok$ by the reverse closure property of $Ok$. Finally, for (iv) $\sigma \in \{\sigma : \exists \tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-, \tau \subseteq \sigma \subseteq \pi\}$, note that we've already seen in cases (i–iii) that all $\tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-$ are not in $Ok$. So if $\tau \subseteq \sigma \subseteq \pi$, then $\sigma \notin Ok$ by the reverse convexity property of $Ok$. □

In combination with our previous observations, this lemma gives us the soundness of our semantics for $\mathbf{BDL}^-$:

**Theorem 5.2** (Soundness for $\mathbf{BDL}^-$)**.** *For all* $\Gamma \subseteq \mathcal{L}_D$ *and* $A \in \mathcal{L}_D$*,*

$$\Gamma \vdash_{\mathbf{BDL}^-} A \Rightarrow \Gamma \vDash_{\mathbf{BDL}^-} A.$$

*Proof.* By Lemma 3.18, all substitution instances of classical tautologies are true in all $\mathbf{BDL}^-$ models. By Lemma 5.1, (M) and (AGG) are true in all $\mathbf{BDL}^-$ models. (MP) preserves truth in $\mathbf{BDL}^-$ models by a standard argument. And by Theorem 3.10, (RBE) preserves truth in $\mathbf{BDL}^-$ models. □

Next we turn to completeness. For this, we need a few auxiliary concepts. Remember that a *literal* is either an atom $p \in \mathcal{A}$ or the negation $\neg p$ of an atom $p \in \mathcal{A}$. In the following, we'll use $\lambda$ (possibly indexed) as a meta-variable for literals.

Remember further that a formula is in *conjunctive normal form* iff it is a conjunction of disjunction of literals:

**Definition 5.3.** *A formula $A \in \mathcal{L}$ is in* conjunctive normal form *(CNF) iff*

$$A = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})$$

*for literals $\lambda_i^j$ and $f$ a function that gives us the length of the $i$-th conjunct.*

In standard propositional logic, we can show that every formula is equivalent to a formula in CNF. This is the CNF Theorem. We have the following truthmaker version of this result, which essentially says that every formula is *analytically* equivalent to a formula in CNF:

**Theorem 5.4** (Analytic CNF Theorem)**.** *For every formula $A \in \mathcal{L}$, there is a formula $B$ in conjunctive normal form, such that both*

1. $[\![A]\!]^+ = [\![B]\!]^+$ *and*
2. $[\![A]\!]^- = [\![B]\!]^-$.

*Proof.* By induction on the complexity of $A$, as in the standard proof of the CNF Theorem. Note that all the transformations in the standard argument are also analytic equivalences. Note in particular that we have $[\![A \vee (B \wedge C)]\!]^+ = [\![(A \vee B) \wedge (A \vee C)]\!]^+$ and $[\![A \vee (B \wedge C)]\!]^- = [\![(A \vee B) \wedge (A \vee C)]\!]^-$, which essentially relies on the fact that we're using replete contents! □

Literals correspond in a natural way to the "building blocks" of our states, i.e. pairs of the form $(p, x)$, where $p \in \mathcal{A}$ and $x \in \{0, 1\}$:

**Definition 5.5** (Trace states)**.** *Let $\lambda$ be a literal. Then we define:*

$$tr(\lambda) := \begin{cases} (p, 1) & \text{if } \lambda = p \\ (p, 0) & \text{if } \lambda = \neg p \end{cases}.$$

Now, to the completeness argument. The argument proceeds in a more or less standard fashion by means of a canonical model construction via maximally consistent sets.

**Definition 5.6.** *A set of formulas $\Phi$ is consistent with respect to* $\mathbf{BDL}^-$ *iff there is no $A$ such that $\Phi \vdash_{\mathbf{BDL}^-} A$ and $\Phi \vdash_{\mathbf{BDL}^-} \neg A$.*

**Definition 5.7.** *A set of formulas $\Phi$ is maximally consistent with respect to* $\mathbf{BDL}^-$ *iff both of the following hold:*

1. *$\Phi$ is consistent with respect to* $\mathbf{BDL}^-$
2. *for all sets $\Phi' \supseteq \Phi$ that are consistent with respect to* $\mathbf{BDL}^-$, $\Phi = \Phi'$.

By a standard argument, we can show that every consistent set of formulas can be extended to a maximally consistent set:

**Theorem 5.8.** *For any consistent set of formulas $\Phi$ there is a maximally consistent set of formulas $\Phi^*$ such that $\Phi \subseteq \Phi^*$.*

Note that maximally consistent sets have the following canonical properties:

**Lemma 5.9.** *If $\Phi$ is maximally consistent with respect to* $\mathbf{BDL}^-$*, then for all $A$,*

1. *either $A \in \Phi$ or $\neg A \in \Phi$ and never both*
2. *$A \in \Phi \Leftrightarrow \Phi \vdash A$*
3. *$A \notin \Phi \Leftrightarrow \Phi \vdash \neg A$*

A maximally consistent set determines a canonical model as follows:

**Definition 5.10** (Canonical Model)**.** *Let $\Phi$ be a set of formulas that is maximally consistent with respect to* $\mathbf{BDL}^-$*. We define the canonical* $\mathbf{BDL}^-$ *model $\mathcal{M}_\Phi = (\omega_\Phi, Ok_\Phi)$ for $\Phi$ as follows:*

*(i)* $\omega_\Phi(p) = \begin{cases} 1 & \text{if } p \in \Phi \\ 0 & \text{if } p \notin \Phi \end{cases}$

*(ii)* $Ok_\Phi = \{\overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}} : \Phi \nvdash_{\mathbf{BDL}^-} O(\lambda_1 \vee \ldots \vee \lambda_n)\}$

For the remainder of the proof it's useful to note that any state $\sigma$ can be written as $\overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}}$ for suitable literals $\lambda_1, \ldots, \lambda_n$. In the following, we make free use of this observation without further notice.

We now check that $\mathcal{M}_\Phi$ is indeed a $\mathbf{BDL}^-$ model:

**Lemma 5.11.** *For $\Phi$ a maximally consistent set of formulas and $\mathcal{M}_\Phi = (\omega_\Phi, Ok_\Phi)$ as defined in Definition 5.10:*

1. *$\omega_\Phi$ is a valuation,*
2. *$Ok_\Phi$ is reverse convex,*
3. *$Ok_\Phi$ is reverse closed.*

*Proof.*

1. Since for all $p$, either $p \in \Phi$ or $p \notin \Phi$ and never both, we know that $\omega_\Phi$ is a well-defined function from $\mathcal{A}$ to $\{0, 1\}$.
2. To see that $Ok_\Phi$ is reverse convex, suppose that $\tau, \pi \notin Ok_\Phi$ and $\tau \subseteq \sigma \subseteq \pi$. This means that:

   - $\tau = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}}$
   - $\sigma = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n), \ldots, tr(\lambda_{n+m})\}}$
   - $\pi = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n), \ldots, tr(\lambda_{n+m}), \ldots, tr(\lambda_{n+m+k})\}}$

   with

   - $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$

- $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$

To see this note that $\overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}} \notin \{\overline{\{tr(\gamma_1), \ldots, tr(\gamma_n)\}} : \Phi \nvdash O(\gamma_1 \vee \ldots \vee \gamma_n)$, for literals $\gamma_{1 \leq i \leq n}\}$ just in case it's not the case that $\Phi \nvdash_{\mathbf{BDL}^-} O(\lambda_1 \vee \ldots \vee \lambda_n)$, meaning precisely $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$.

We want to show that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m})$. Since $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$ and $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$ we can infer via (AGG) that $\Phi \vdash O((\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda_1 \vee \ldots \vee \lambda_{n+m+k}))$. It is somewhat tedious but possible to show in AC that $(\lambda_1 \vee \ldots \vee \lambda_{n+m}) \wedge (\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$ is analytically equivalent to $(\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$. By a single application of (RBM) (see §2) we get $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m})$, meaning $\sigma \notin Ok_\Phi$ as desired.

3. To show that $Ok_\Phi$ is reverse closed, suppose that $\sigma, \tau \notin Ok_\Phi$, meaning:

   - $\sigma = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}}$
   - $\tau = \overline{\{tr(\lambda'_1), \ldots, tr(\lambda'_m)\}}$

   with

   - $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$
   - $\Phi \vdash O(\lambda'_1 \vee \ldots \vee \lambda'_m)$.

   We want to show that that $\sigma \cup \tau \notin Ok_\Phi$, meaning that

   $$\sigma \cup \tau = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n), tr(\lambda'_1), \ldots, tr(\lambda'_m)\}}$$

   is such that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n \vee \lambda'_1 \vee \ldots \vee \lambda'_m)$. Since we know that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$ and $\Phi \vdash O(\lambda'_1 \vee \ldots \vee \lambda'_m)$, we can conclude via (AGG) that $\Phi \vdash O((\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda'_1 \vee \ldots \vee \lambda'_m))$. Since we know that $\vdash O(A \wedge B) \to O(A \vee B)$, we can infer that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n \vee \lambda'_1 \vee \ldots \vee \lambda'_m)$, as desired.

□

Note that $\Phi$ being maximally consistent doesn't play a role for the fact that $Ok_\Phi$ is reverse convex and reverse closed. It does play a role, however, in showing that $\mathcal{M}_\Phi$ has the following important property:

**Lemma 5.12** (Truth lemma)**.** *If $\Phi$ is maximally consistent, then*

$$\mathcal{M}_\Phi \vDash A \Leftrightarrow A \in \Phi.$$

*Proof.* By induction on the complexity of the members of $A$:

1. If $p \in \Phi$, then $\omega_\Phi(p) = 1$ by definition, and so $\mathcal{M}_\Phi \vDash_{\mathbf{BDL}^-} p$ as desired. If $p \notin \Phi$, then $\omega_\phi(p) = 0$ and so $\mathcal{M}_\Phi \nvDash p$ as desired.
2. If $\neg A \in \Phi$, then $A \notin \Phi$, since $\Phi$ is consistent. Hence $\mathcal{M}_\Phi \nvDash A$ by the induction hypothesis and so $\mathcal{M}_\Phi \vDash_{\mathbf{BDL}^-} \neg A$. If $\neg A \notin \Phi$, then $A \in \Phi$, since $\Phi$ is maximally consistent. Hence $\mathcal{M}_\Phi \vDash_{\mathbf{BDL}^-} A$ and so $\mathcal{M}_\Phi \nvDash \neg A$ as desired.

3. If $A \wedge B \in \Phi$, then, since $A \wedge B \vdash A$ and $A \wedge B \vdash B$ and $\Phi$ is maximally consistent, we get that $A, B \in \Phi$. So $\mathcal{M}_\Phi \models A$ and $\mathcal{M}_\Phi \models B$ by the induction hypothesis and so $\mathcal{M}_\Phi \models A \wedge B$ as desired. If $A \wedge B \notin \Phi$, then $\neg(A \wedge B) \in \Phi$ since $\Phi$ is maximally consistent. Next we show that $\neg A \in \Phi$ or $\neg B \in \Phi$. For if both $\neg A \notin \Phi$ and $\neg B \notin \Phi$, then, since $\Phi$ is maximally consistent, we'd have $A, B \in \Phi$. But then we'd get $\Phi \vdash A \wedge B$, and, since $\neg(A \wedge B) \in \Phi$, $\Phi \vdash \neg(A \wedge B)$, making $\Phi$ inconsistent. So $\neg A \in \Phi$ or $\neg B \in \Phi$. So either $A \notin \Phi$ of $B \notin \Phi$. So either $\mathcal{M}_\Phi \not\models A$ or $\mathcal{M}_\Phi \not\models B$. Hence $\mathcal{M}_\Phi \not\models A \wedge B$ as desired.

4. The cases for $A \vee B \in \Phi$ and $A \vee B \notin \Phi$ are analogous to the previous case.

5. The main cases are (a) $OA \in \Phi$ and (b) $OA \notin \Phi$.

   (a) Suppose that $OA \in \Phi$. Then, of course, $\Phi \vdash OA$. Now by the analytic CNF Theorem 5.4, there is a $B$ in CNF, such that $[\![A]\!]^+ = [\![B]\!]^+$ and $[\![A]\!]^- = [\![B]\!]^-$. By Theorem 3.10, we get $\vdash_{AC} A \Leftrightarrow_A B$ and so by (RBE) $\Phi \vdash OB$. Let

   $$B = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}),$$

   where $f$ is a function that gives us the length of the $i$-th conjunct of $B$. By (M), we get

   $$\Phi \vdash O(\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)})$$

   for all $1 \leq i \leq n$. By definition of $Ok_\Phi$ this means that $\overline{\{tr(\lambda_i^1), \ldots, tr(\lambda_i^{f(i)})\}} \notin Ok_\Phi$. It is easily checked that $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- = \{\overline{\{tr(\lambda_i^1), \ldots, tr(\lambda_i^{f(i)})\}}\}$ and so $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- \cap Ok_\Phi = \emptyset$ for all $1 \leq i \leq n$. By repeated application of Lemma 5.1, we get

   $$[\![(\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})]\!]^- \cap Ok_\Phi = \emptyset,$$

   which just means that $\mathcal{M}_\Phi \models O((\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}))$. Since

   $$B = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})$$

   and $[\![A]\!]^- = [\![B]\!]^-$, this implies that $\mathcal{M}_\Phi \models OA$.

   (b) Suppose $OA \notin \Phi$. Then, since $\Phi$ is maximally consistent, $\Phi \nvdash OA$. Again, by the analytic CNF Theorem 5.4 and Theorem 3.10, there's a $B$ such that $\vdash_{AC} A \Leftrightarrow_A B$. We can conclude that also $\Phi \nvdash OB$. For if $\Phi \vdash OB$, then by (RBE) we'd get $\Phi \vdash OA$ and so $OA \in \Phi$, since $\Phi$ is maximally consistent. Contradiction. Hence $\Phi \nvdash OB$.Now let again

   $$B = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}),$$

   where $f$ is again a function that gives us the length of the $i$-th conjunct of $B$. Next we conclude that $\Phi \nvdash O(\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)})$, for some $1 \leq i \leq n$.

For if $\Phi \vdash O(\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)})$ for all $1 \leq i \leq n$, we'd get $\Phi \vdash OB$ by (AGG). Since $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- = \{\{tr(\lambda_i^1), \ldots, tr(\lambda_i^{f(i)})\}\}$ and $Ok_\Phi = \{\{tr(\lambda_1), \ldots, tr(\lambda_n)\} : \Phi \nvdash O(\lambda_1 \vee \ldots \vee \lambda_n)\}$, we can conclude that $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- \cap Ok_\Phi \neq \emptyset$. By Lemma 5.1, we can conclude that

$$[\![(\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})]\!]^- \cap Ok_\Phi \neq \emptyset,$$

which just means that $\mathcal{M}_\Phi \nvDash O((\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}))$. And since

$$B = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})$$

and $[\![A]\!]^- = [\![B]\!]^-$, this implies that $\mathcal{M}_\Phi \nvDash OA$.

□

The rest of the proof is routine:

**Lemma 5.13.** *If $\Phi \nvdash A$, then $\Phi \cup \{\neg A\}$ is consistent.*

**Theorem 5.14** (Completeness for $\mathbf{BDL}^-$)**.** *For all $\Phi \subseteq \mathcal{L}_D$ and $A \in \mathcal{L}_D$,*

$$\Phi \vDash_{\mathbf{BDL}^-} A \Rightarrow \Phi \vdash_{\mathbf{BDL}^-} A.$$

*Proof.* We prove the contrapositive. So suppose that $\Phi \nvdash A$. Then $\Phi \cup \{\neg A\}$ is consistent by Lemma 5.13. Extend $\Phi \cup \{\neg A\}$ to a maximally consistent set $(\Phi \cup \{\neg A\})^* \supseteq \Phi \cup \{\neg A\}$ using Theorem 5.8. Consider $\mathcal{M}_{(\Phi \cup \{\neg A\})^*}$. By Lemma 5.12, we get $\mathcal{M}_{(\Phi \cup \{\neg A\})^*} \vDash \phi$, for all $\phi \in \Phi$, and $\mathcal{M}_{(\Phi \cup \{\neg A\})^*} \vDash \neg A$. So, in particular, $\mathcal{M}_{(\Phi \cup \{\neg A\})^*} \vDash \phi$, for all $\phi \in \Phi$, and $\mathcal{M}_{(\Phi \cup \{\neg A\})^*} \nvDash A$, giving us that $\Phi \nvDash A$.

□

This concludes our proof of the soundness and completeness of $\mathbf{BDL}^-$.

As advertised, the soundness and completeness of **BDL** follows as a corollary from this result:

**Corollary 5.15** (Soundness and Completeness of **BDL**)**.** *For all $\Phi \subseteq \mathcal{L}_D$ and $A \in \mathcal{L}_D$,*

$$\Phi \vDash_{\mathbf{BDL}} A \Leftrightarrow \Phi \vdash_{\mathbf{BDL}} A.$$

*Proof.* We have $\Phi \vdash_{\mathbf{BDL}} A$ iff $\Phi \cup \{(O(A \vee B) \wedge O\neg A) \to OB : A, B \in \mathcal{L}\} \vdash_{\mathbf{BDL}^-} A$. By the soundness and completeness for $\mathbf{BDL}^-$, we have that $\Phi \cup \{(O(A \vee B) \wedge O\neg A) \to OB : A, B \in \mathcal{L}\} \vdash_{\mathbf{BDL}^-} A$ iff $\Phi \cup \{(O(A \vee B) \wedge O\neg A) \to OB : A, B \in \mathcal{L}\} \vDash_{\mathbf{BDL}^-} A$. Finally, by Theorem 4.8, we know that $\Phi \cup \{(O(A \vee B) \wedge O\neg A) \to OB : A, B \in \mathcal{L}\} \vDash_{\mathbf{BDL}^-} A$ iff $\Phi \vDash_{\mathbf{BDL}} A$.

To see the latter, first note that, by Definition 4.9 $\mathcal{M} = (\omega, Ok)$ is a **BDL** model iff $\mathcal{M}$ is a $\mathbf{BDL}^-$ model in which for all finite sets of finite states $\Sigma, \Delta$, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$, then $conv(\overline{\Delta}) \cap Ok = \emptyset$. By Theorem

4.8, this means that $\mathcal{M}$ is a **BDL** model iff $\mathcal{M} \vDash (O(A \vee B) \wedge O\neg A) \rightarrow B$, for all $A, B \in \mathcal{L}$.

Now suppose that $\Phi \cup \{(O(A \vee B) \wedge O\neg A) \rightarrow OB : A, B \in \mathcal{L}\} \vDash_{\mathbf{BDL}^-} A$, i.e. for all $\mathbf{BDL}^-$ models $\mathcal{M}$, if $\mathcal{M} \vDash C$, for all $C \in \Phi \cup \{(O(A \vee B) \wedge O\neg A) \rightarrow OB : A, B \in \mathcal{L}\} \vdash$, then $\mathcal{M} \vDash A$. But we've just seen that $\mathcal{M}$ is a **BDL** model iff $\mathcal{M} \vDash (O(A \vee B) \wedge O\neg A) \rightarrow B$, for all $A, B \in \mathcal{L}$. So it follows that for all **BDL** models $\mathcal{M}$, if $\mathcal{M} \vDash C$, for all $C \in \Phi$, then $\mathcal{M} \vDash A$—or, in other words, $\Phi \vDash_{\mathbf{BDL}} A$.

Conversely, suppose that $\Phi \vDash_{\mathbf{BDL}} A$, i.e. for all **BDL** models $\mathcal{M}$, if $\mathcal{M} \vDash C$, for all $C \in \Phi$, then $\mathcal{M} \vDash A$. But since a $\mathbf{BDL}^-$ model $\mathcal{M}$ is a **BDL** model iff $\mathcal{M} \vDash (O(A \vee B) \wedge O\neg A) \rightarrow B$, for all $A, B \in \mathcal{L}$, we can infer that for all $\mathbf{BDL}^-$ models, $\mathcal{M}$, if $\mathcal{M} \vDash C$, for all $C \in \Phi \cup \{(O(A \vee B) \wedge O\neg A) \rightarrow OB : A, B \in \mathcal{L}\} \vdash$, then $\mathcal{M} \vDash A$—as desired.

□

# 6 Discussion

After having established our main result, we take this conclusion to discuss some limitations and possible extensions of our work.

## 6.1 Modularity

In this paper, we developed a sound and complete semantics for **BDL** and, along the way, for $\mathbf{BDL}^-$. This was our main goal, and we went straight for the desired result. To obtain the result, we postulated two conditions on the set $Ok$ of admissible states, namely reverse closure and reverse complexity. A natural question to ask at this point is what happens if we drop these two conditions?[23] Does our semantics allow for a kind of modularity where certain conditions on the $Ok$-set correspond to the validity of certain formulas?

Observe that in the proof of the soundness lemma (Lemma 5.1), we needed both conditions, reverse closure and reverse convexity, to establish the semantic fact that corresponds to the validity of (AGG). In fact, if we drop either reverse closure and reverse convexity, (AGG) will no longer be valid.

To see this, first consider the "model" $\mathcal{M} = (\omega, Ok)$ with $Ok = \{\sigma : \{(p, 0), (q, 0)\} \subseteq \sigma\}$ where, for concreteness sake, we let $\omega(p) = 0$ for all $p \in \mathcal{A}$. This "model" doesn't satisfy reverse closure since $\{(p, 0)\}, \{(q, 0)\} \notin Ok$ but $\{(p, 0)\} \cup \{(q, 0)\} = \{(p, 0), (q, 0)\} \in Ok$. So, strictly speaking, $\mathcal{M}$ is not a model, since we require that models be reverse closed. But let's pretend for the moment that we redefine everything to allow for structures like $\mathcal{M}$ as models. Note that $\mathcal{M}$ *does* satisfy reverse convexity.[24] By applying our definition for $\mathcal{M} \vDash OA$ in this case, we get that $\mathcal{M} \vDash Op$ and $\mathcal{M} \vDash Oq$, since $[\![p]\!]^- = \{\{(p, 0)\}\} \cap Ok = \emptyset$

[23]We'd like to thank an anonymous referee for raising this question.

[24]*Proof*: For proof by contradiction, assume that $\sigma, \tau \notin Ok$ but there is a $\pi$ with $\sigma \subseteq \pi \subseteq \tau$ with $\pi \in Ok$—meaning that reverse convexity is violated. Then, since $\pi \in Ok$, $\{(p, 0), (q, 0)\} \subseteq \pi$. So, since $\pi \subseteq \tau$, $\{(p, 0), (q, 0)\} \subseteq \tau$. But then $\tau \in Ok$. Contradiction. So $Ok$ is reverse convex.

and $[\![q]\!]^- = \{\{(q,0)\}\} \cap Ok = \emptyset$. At the same time, however, we have that $\mathcal{M} \nvDash O(p \wedge q)$, since $[\![p \wedge q]\!]^- = \{\{(p,0),(q,0)\}\} \cap Ok = \{\{(p,0),(q,0)\}\} \neq \emptyset$. So we get that $\mathcal{M} \nvDash Op \wedge Oq \to O(p \wedge q)$—meaning that if we allow non reverse closed models like $\mathcal{M}$, (AGG) wouldn't be valid.

A similar argument can be given for reverse convexity. Take the model $\mathcal{M} = \{\omega, Ok\}$ with $\omega$ like above and $Ok = \{\{(q,0)\},\{(p,0),(q,0)\}\}$. In this model reverse closure is satisfied but reverse convexity fails. To see the former, note that there are no $\sigma, \tau \notin Ok$ such that $\tau \cup \sigma \in Ok$. To see the latter, note that $\{(p,0)\} \notin Ok$ and $\{(p,0),(q,0),(r,0)\} \notin Ok$ but $\{(p,0)\} \subseteq \{(p,0),(q,0)\} \subseteq \{(p,0),(q,0),(r,0)\}$ and $\{(p,0),(q,0)\} \in Ok$. Now note that $\mathcal{M} \vDash Op$, since $[\![p]\!]^- = \{\{(p,0)\}\} \cap Ok = \emptyset$, and note that $\mathcal{M} \vDash O(p \vee q \vee r)$, since $[\![p \vee q \vee r]\!]^- = \{\{(p,0),(q,0),(r,0)\}\} \cap Ok = \emptyset$. However, we have $\mathcal{M} \nvDash O(p \wedge (p \vee q \vee r))$. To see this, first note that $\{(p,0)\} \in [\![p \wedge (p \vee q \vee r)]\!]^-$ and $\{(p,0),(q,0),(r,0)\} \in [\![p \wedge (p \vee q \vee r)]\!]^-$. Since $\{(p,0)\} \subseteq \{(p,0),(q,0)\} \subseteq \{(p,0),(q,0),(r,0)\}$ and $[\![p \wedge (p \vee q \vee r)]\!]^-$ is convex, it follows that $\{(p,0),(q,0)\} \in [\![p \wedge (p \vee q \vee r)]\!]^-$. Therefore, $[\![p \wedge (p \vee q \vee r)]\!]^- \cap Ok \neq \emptyset$ and so $\mathcal{M} \nvDash O(p \wedge (p \vee q \vee r))$.

To sum up, if we allow our models to violate either the reverse closure or the reverse convexity of $Ok$, (AGG) will turn out invalid. Since these two conditions together do entail the validity of (AGG), we get that the validity of (AGG) is, in our semantics, equivalent to the conjunction of reverse closure and reverse convexity.

Finally, note that, in order to establish the semantic fact that corresponds to the validity of (M), *no* conditions on $Ok$ were needed—the fact follows from the semantic clause for the $O$-operator and basic set-theory. This means that even if we drop reverse closure and reverse convexity, (M) will still be valid on the resulting semantics. In order to invalidate (M), we need to tinker with the semantic clause for $O$. It's an open question to determine whether there's a more general form of our semantics, with a different clause for $O$, where the validity of (M) is equivalent to a condition on the structure.

### 6.2 Goble's Desiderata

In [11, p. 318] Goble claims that **BDL** meets all three desiderata for a logic for normative conflicts. As we have already indicated at the end of section 3, we can now formally prove this. Let us first reiterate Goble's desiderata:

**Consistent Conflicts.** At least some normative conflicts should be consistent, i.e. we can have $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$ but $OA_1, \ldots, OA_n \nvdash \bot$.

**No Deontic Explosion.** Normative conflicts should not result in *deontic explosion*, i.e. we can have $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$ but $OA_1, \ldots, OA_n \nvdash OB$.

**Minimal Deontic Laws.** Certain minimal laws of deontic logic should be validated:

| | | |
|---|---|---|
| (DDS) | $O(A \vee B), O\neg A \vdash OB$ | ('deontic disjunctive syllogism'). |
| (M) | $O(A \wedge B) \vdash OB$ | ('monotonicity'). |
| (AGG) | $OA, OB \vdash O(A \wedge B)$ | ('aggregation'). |

Put in semantic terms, these desiderata can be expressed by the following theorem:

**Theorem 6.1.** *For some $A_1, \ldots, A_n, \ldots, B_1, \ldots, B_m, C$, there is a **BDL** model $\mathcal{M}$ such that:*[25]

**(RBM)** $\vDash \neg(A_1 \wedge \ldots \wedge A_n)$ *and* $\mathcal{M} \vDash OA_1, \ldots, OA_n$

**(NDE)** $\vDash \neg(B_1 \wedge \ldots \wedge B_m)$ *and* $\mathcal{M} \vDash OB_1, \ldots, OB_m$ *and* $\mathcal{M} \nvDash OC$

**(DL1)** $O(A \vee B), O\neg A \vDash OB$

**(DL2)** $O(A \wedge B) \vDash OB$

**(DL3)** $OA, OB \vDash O(A \wedge B)$

*Proof.* Given that we've already established the soundness and completeness of **BDL**, this proof turns out to be very simple. (DL1), (DL2) and (DL3) follow from **BDL**'s soundness. Since (NDE) implies (RBM) it suffices to prove (NDE). We construct a **BDL** model $\mathcal{M}$ such that $\mathcal{M} \vDash Op \wedge O\neg p$ but $\mathcal{M} \nvDash Oq$. Since our background logic is classical, we have that $\vDash \neg(p \wedge \neg p)$, hence such a model would indeed give us (NDE).

Let $\mathcal{M} = (\omega, Ok)$ with any classical valuation $\omega$ (for concreteness sake, let $\omega(p) = 0$ for all $p \in \mathcal{P}$), and $Ok = \{\sigma : \sigma \text{ is a state}\} \setminus \{\{(p,1)\}, \{(p,0)\}, \{(p,0),(p,1)\}\}$. Since there are only three states not contained in $Ok$, it's quite easily checked that $Ok$ is reverse convex and reverse closed. So $\mathcal{M}$ is a **BDL**$^-$ model. To show that $\mathcal{M}$ is also a **BDL** model, following Definition 4.9, we have to establish that:

- For all finite sets of finite states $\Sigma, \Delta$, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$, then $conv(\overline{\Delta}) \cap Ok = \emptyset$.

There are, in fact, only 8 finite sets of states whose convex closures have an empty intersection with $Ok$, namely the subsets of $\{\{(p,1)\}, \{(p,0)\}, \{(p,0),(p,1)\}\}$. But it's easily checked that for each of these, the condition is satisfied. Here just one example: take $\Sigma = \{\{(p,1)\}\}$ and $\Delta = \{\{(p,0)\}\}$. We have:

$$conv(\overline{\Sigma} \circ \overline{\Delta}) = conv(\overline{\{\{(p,1)\}\}} \circ \overline{\{\{(p,0)\}\}}) = conv(\{(p,0),(p,1)\}) = \{(p,0),(p,1)\} \cap Ok = \emptyset$$

$$conv(\Sigma) = con(\{\{(p,1)\}\}) = \{\{(p,1)\}\} \cap Ok = \emptyset$$

$$conv(\overline{\Delta}) = conv(\overline{\{\{(p,0)\}\}}) = \{\{(p,1)\}\} \cap Ok = \emptyset$$

Hence $\mathcal{M}$ is a **BDL** model. Now, it's easily checked that:

$$[\![p]\!]^- = conv([p]^-) = conv(\{\{(p,0)\}\}) = \{\{(p,0)\}\} \cap Ok = \emptyset,$$

and hence $\mathcal{M} \vDash Op$, as well as:

$$[\![\neg p]\!]^- = conv([p]^-) = conv(\{\{(p,1)\}\}) = \{\{(p,1)\}\} \cap Ok = \emptyset,$$

and hence $\mathcal{M} \vDash O\neg p$. At the same time, we have that

$$[\![\neg q]\!]^- = conv([q]^-) = conv(\{\{(q,0)\}\}) = \{\{(q,0)\}\} \cap Ok = \{\{(q,0)\}\} \neq \emptyset,$$

and hence $\mathcal{M} \nvDash Oq$, as desired. □

[25] For the sake of completeness, we state the theorem in full length here.

### 6.3 Permissions in BDL and BDL$^{-}$

As we mentioned in the introduction, one of the starting points for our paper was Fine's truthmaker semantics for permission statements [7, p. 335]. Fine's idea was that in a truthmaker setting, we can distinguish a set of admissible states (our $Ok$) and then say:

- $PA$ is true iff every state that's a truthmaker for $A$ is admissible.

Using the same notion of admissible states that Fine uses to model

- $OA$ is true iff no falsemaker of $A$ is admissible.

We showed that this semantic clause allows us to provide sound and complete semantics for the Goble's CTDL **BDL**—solving an important open problem in deontic logic.

So, effectively, what we did is that we used a framework for the semantics of permission statements in order to define a semantics for obligation statements. Note that we didn't introduce any new semantic components to the existing framework to interpret obligation. Indirectly (read "semantically"), we obtained a notion of obligation from a concept of permission—*from permission to obligation.*

A natural question to ask at this point is how the logic of permission sketched by Fine and the logic of obligation studied in this paper interact. In order to answer this question, let's first make the above a little bit more precise. First, we add a permission operator $P$ to our language, allowing for permission statements of the form $PA$, where $A \in \mathcal{L}$. We then extend our semantic clauses by saying that for a model $\mathcal{M} = (\omega, Ok)$:

(vi) $\mathcal{M} \vDash PA$ iff $[\![A]\!]^+ \subseteq Ok$.

Note that we're reading "truthmaker" in Fine's semantic clause to mean *replete* truthmaker.[26] So what's the deontic logic that results from this?

A first observation we can make about the interaction of permission and obligation in this setting is that we get

$$OA \vDash \neg P \neg A.$$

The proof is straight-forward: Suppose that $\mathcal{M} \vDash OA$. Then $[\![A]\!]^- \cap Ok = \emptyset$. Since $[\![A]\!]^- = [\![\neg A]\!]^+$, we can infer that $[\![\neg A]\!]^+ \nsubseteq Ok$, i.e. $\mathcal{M} \nvDash P\neg A$ and so $\mathcal{M} \vDash \neg P \neg A$.

The converse, however, doesn't hold:

$$\neg P \neg A \nvDash OA.$$

To see this, consider any model $\mathcal{M} = (\omega, Ok)$ with $Ok = \{\sigma : \{(p, 0)\} \subseteq \sigma\}$. We get that $\mathcal{M} \vDash \neg P \neg (p \wedge q)$. To see this, note that $\{(q, 0)\} \in [\![\neg(p \wedge q)]\!]^+$ but $\{(q, 0)\} \notin Ok$. So $[\![\neg(p \wedge q)]\!]^+ \nsubseteq Ok$, meaning $\mathcal{M} \nvDash P\neg(p \wedge q)$ and so $\mathcal{M} \vDash$

[26] There are other options, of course. See, for example, [3].

$\neg P \neg (p \wedge q)$. But, at the same time, we have that $\{(p,0)\} \in [\![\neg(p \wedge q)]\!]^+ = [\![p \wedge q]\!]^-$ and $\{(p,0)\} \in Ok$. So $\mathcal{M} \not\vDash O(p \wedge q)$.

It is perhaps interesting to observe further that in the present setting, obligation doesn't entail permission:

$$OA \not\vDash PA.$$

To see this, take any model $\mathcal{M} = (\omega, Ok)$ with $Ok = \{\sigma : (q,1) \subseteq \sigma\}$. In such a model, we have $\mathcal{M} \vDash Op$, since $[\![p]\!]^- = \{(p,0)\} \cap Ok = \emptyset$. But we don't have $\mathcal{M} \vDash Pp$, since $[\![p]\!]^+ \not\subseteq Ok$. If all the falsemakers of A are inadmissible, that doesn't mean that all its truthmakers are admissible.

Finally, we may ask ourselves about the logic of permission itself. As Fine [7, p. 335] points out, a nice feature of the present concept of permission is that (using a suitable notion of truthmakers), it gives us the so-called *principle of free choice permission*: $P(A \vee B)$ is logically equivalent to $P(A) \wedge P(B)$. To validate this principle on the present approach, where we read "truthmaker" in Fine's clause to mean replete truthmaker, we need to postulate the following two conditions on the set $Ok$ of admissible states:

**(Closure)** If $\sigma, \tau \in Ok$, then $\sigma \cup \tau \in Ok$,

**(Convexity)** If $\sigma, \tau \in Ok$ and $\sigma \subseteq \pi \subseteq \tau$, then $\pi \in Ok$.

Given these two conditions on the $Ok$ set, we can easily establish that:

$$\vDash P(A \vee B) \leftrightarrow PA \wedge PB.$$

For suppose that $\mathcal{M} = (\omega, Ok)$ is a model in which $Ok$ is closed and convex in the above sense. Suppose further that $\mathcal{M} \vDash P(A \vee B)$, i.e. $[\![A \vee B]\!]^+ \subseteq Ok$. $[\![A \vee B]\!]^+ = conv([A]^+ \cup [B]^+ \cup ([A]^+ \circ [B]^+))$. Since $[A]^+ \cup [B]^+ \subseteq conv([A]^+ \cup [B]^+ \cup ([A]^+ \circ [B]^+))$, we get $[A]^+ \cup [B]^+ \subseteq Ok$. By set-theory, we get $[A]^+ \subseteq Ok$ and $[B]^+ \subseteq Ok$. But since taking the convex closure of a set is monotonic, meaning if $\Sigma \subseteq \Delta$, then $conv(\Sigma) \subseteq conv(\Delta)$ (cf. Lemma 3.7), we get that $conv([A]^+) = [\![A]\!]^+ \subseteq conv(Ok)$ and $conv([B]^+) = [\![B]\!]^+ \subseteq conv(Ok)$. But since $Ok$ is convex by assumption, we have $conv(Ok) = Ok$ and we get $\mathcal{M} \vDash PA$ and $\mathcal{M} \vDash PB$, as desired.

Conversely, suppose that $\mathcal{M} \vDash PA$ and $\mathcal{M} \vDash PB$, i.e. $[\![A]\!]^+ \subseteq Ok$ and $]\!]B]\!]^+ \subseteq Ok$. Since $[A]^+ \subseteq [\![A]\!]^+$, we get that $[A]^+ \subseteq Ok$ and, analogously, $[B]^+ \subseteq Ok$. By set-theory, it follows that $[A]^+ \cup [B]^+ \subseteq Ok$. Next, we establish that $[A \wedge B]^+ = [A]^+ \circ [B]^+ \subseteq Ok$. To show this, we need the closure of $Ok$ under unions. Suppose that $\sigma \in [A]^+ \circ [B]^+$. This means, by definition, that there is a $\tau \in [A]^+$ and a $\pi \in [B]^+$ such that $\sigma = \tau \cup \pi$. Since $\tau \in [A]^+$ and $\pi \in [B]^+$, we have $\tau, \pi \in Ok$—for, as we've just established, $[A]^+ \subseteq Ok$ and $[B]^+ \subseteq Ok$. But since we've assumed that $Ok$ is closed under $\cup$, we get that $\tau \cup \pi = \sigma \in Ok$, as desired. So, we also have $[A \wedge B]^+ \subseteq Ok$, which together with the previously established fact that $[A]^+ \cup [B]^+ \subseteq Ok$, gives us $[A]^+ \cup [B]^+ \cup [A \wedge B]^+ \subseteq Ok$. By the monotonicity of taking convex closures, it follows that $conv([A]^+ \cup [B]^+ \cup [A \wedge B]^+) = [\![A \vee B]\!]^+ \subseteq conv(Ok) = Ok$, meaning $\mathcal{M} \vDash P(A \vee B)$ as claimed.

This is, in our opinion, a desirable feature of the present concept of permission, since the principle of free choice permission is notoriously difficult to model semantically (see, e.g., [13, p. 214–17]). Demanding that $Ok$ be closed under $\cup$ and convex, however, also has some unexpected consequences for *obligation.* For example, the move validates:

$$O(A \vee B) \rightarrow (OA \vee OB).^{27}$$

To see this, assume that we have a model $\mathcal{M} = (\omega, Ok)$ where $Ok$ is closed and convex and $\mathcal{M} \vDash O(A \vee B)$, i.e. $[\![A \vee B]\!]^- \cap Ok = \emptyset$. Now assume, for proof by contradiction, that both $\mathcal{M} \nvDash OA$ and $\mathcal{M} \nvDash OB$, i.e. $[\![A]\!]^- \cap Ok \neq \emptyset$ and $[\![B]\!]^- \cap Ok \neq \emptyset$. This means that there's a $\sigma \in [\![A]\!]^-$ with $\sigma \in Ok$ and there's a $\tau \in [\![B]\!]^-$ with $\tau \in Ok$. Since $\sigma \in [\![A]\!]^-$ and $\tau \in [\![B]\!]^-$, $\sigma \cup \tau \in [\![A \vee B]\!]^-$. And since $\sigma, \tau \in Ok$, $\sigma \cup \tau \in Ok$ by closure. But then $[\![A \vee B]\!]^- \cap Ok \neq \emptyset$. Contradiction. Hence we have $\mathcal{M} \vDash OA$ or $\mathcal{M} \vDash OB$

What the previous example illustrates is that, given how we defined our concept of obligation indirectly via a concept of permission, there can be unexpected interactions between permission and obligation. It would be interesting to further investigate the deontic logic that we were only able to sketch here, but we'll have to leave this for further research.

**Acknowledgements** For very helpful comments on earlier versions of this paper we would like to thank O. Foisch, Hannes Leitgeb, Frederik Van De Putte and the audiences at Trends in Logic (Lublin, 2017), at Hyperintensional Logics and Truthmaker Semantics (Ghent, 2017) and at the third PIOTR workshop at the University of Bayreuth. This research was partly financed the the DFG as part of the PIOTR-Project at the University of Bayreuth.

## References

[1] Angell, R.B. 1977. "Three Systems of First Degree Entailment." *Journal of Symbolic Logic* 47, 147.

[2] Angell, R.B. 1989. "Deducibility, Entailment and Analytic Containment." In: *Directions in Relevant Logic*, edited by Richard Sylvan and Jean Norman, 119–144. Kluwer.

[3] Anglberger, Albert & Korbmacher, Johannes & Faroldi, Federico. 2016. "An Exact Truthmaker Semantics for Permission and Obligation" In: Olivier Roy, Allard Tamminga, Malte Willer (eds.) *Deontic Logic and Normative Systems: DEON 2016*, College Publications, 2016, 16–32.

[4] Carnap, Rudolf. 1956. *Meaning and Necessity.* Second Edition. Chicago, IL: University of Chicago Press.

[27] We'd like to thank an anonymous referee for pointing this out to us.

[5] Correia, Fabrice. 2004. "Semantics for Analytic Containment." In: *Studia Logica* 77: 87–104.

[6] Ferguson, Thomas. 2016. "Faulty Belnap Computers and Subsystems of FDE." In: *Journal Logic and Computation* 26(5): 1617–636.

[7] Fine, Kit. 2014. "Permission and Possible Worlds." *Dialectica* 68(3): 317–36.

[8] Fine, Kit. 2016. "Angellic Content." In: *Journal of Philosophical Logic* 42(2): 199–226.

[9] Fine, Kit. 2017. "Truthmaker Semantics." In *A Companion to the Philosophy of Language*, Second Edition, edited by Bob Hale, Crispin Wright, and Alexander Miller, New York, NY: Wiley, 556–577.

[10] Van Fraassen, Bas. 1969. "Facts and Tautological Entailments." *The Journal of Philosophy* 66: 477–487.

[11] Goble, Lou. 2013. "Prima Facie Norms, Normative Conflicts, and Dilemmas" In: Dov Gabbey, John Horty, Xavier Parent, Ron van der Meyden, Leendert van der Torre (eds.) *Handbook of Deontic Logic and Normative Systems*, College Publications 2013, pp. 241–351.

[12] Goble, Lou. 2014. "Deontic logic (adapted) for normative conflicts," in Logic Journal of the IGPL, vol. 22, no. 2: 206–235.

[13] Hansson, Sven Ove. 2013. "The Varieties of Permission." In: Dov Gabbey, John Horty, Xavier Parent, Ron van der Meyden, Leendert van der Torre (eds.) *Handbook of Deontic Logic and Normative Systems*, College Publications 2013, 195–240.

[14] Sartre, Jean-Paul, 1957/1946, "Existentialism is a Humanism," Trans, Philip Mairet, in Walter Kaufmann (ed.), *Existentialism from Dostoevsky to Sartre*, New York: Meridian, 287–311.

[15] Straßer, Christian and Mathieu Beirlaen. 2012. "An Andersonian Deontic Logic with Contextualized Sanctions." In: Thomas Ågotnes, Jan Broersen, and Dag Elgesem (eds.) *Deontic Logic in Computer Science. 11th International Conference, DEON 2012, Bergen, Norway, July 16-18, 2012. Proceedings*, Springer 2012, 151–69.

[16] Straßer, Christian. 2014 *Adaptive Logics for Defeasible Reasoning.* Springer.

[17] Yablo, Steven. 2014. *Aboutness.* Princeton, NJ: Princeton University Press.