Tableau for the logic of exact entailment

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Exact truthmaker semantics

Tableaux for exact entailment

The Jago-Fine characterization theorem

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The logic of exact entailment

- A state s is an *exact truthmaker (falsemaker)* for φ iff s necessitates φ's truth (falsity) and is wholly relevant to it (Fine 2017).

- Γ *exactly entails* φ iff for every s, if s is an exact truthmaker for each $\psi \in \Gamma$, then s is an exact truthmaker of φ (Fine and Jago 2017).

Tableaux[1]

- Tableaux are standardly based on satisfiability checking:
  - Γ classically entails φ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.
- In exact truthmaker semantics, *every* Γ is satisfiable:
  - For each Γ there is a model $\mathbf{M}$ with a state s such that s is an exact truthmaker for each $\psi \in \Gamma$.
- We base our tableaux on directly semantic reasoning about exact truthmaking.

[1] "Tableau" is singular, "Tableaux" is plural. Both are pronounced the same.

Conjunction

- φ and ψ jointly exactly entail $\varphi \wedge \psi$.
- But $\varphi \wedge \psi$ does not, in general, exactly entail either φ or ψ.
- Strangely enough, φ and $\varphi \wedge \psi \wedge \chi$ jointly exactly entail both $\varphi \wedge \psi$ and $\varphi \wedge \chi$ ("premise convexity").

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Syntax

- Our language has just the connectives $\neg, \wedge, \vee$.
- $\mathcal{P}$ is the set of propositional variables.
- A *literal* is a propositional variable or the negation of one:

$$\Lambda := \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\}.$$

- $\bigwedge\{\varphi_1, \ldots, \varphi_n\} = (\varphi_1 \wedge (\varphi_2 \wedge \ldots \wedge \varphi_n) \ldots)$

Models

A *model* is a structure $\mathbf{M} = \langle S, \sqsubseteq, V^+, V^- \rangle$, such that:

- S is a set of entities ('states'),
- $\sqsubseteq$ is a partial order on S ('parthood'), such that every pair $s, t \in S$ has a least upper bound $s \sqcup t \in S$ ('fusion'), and
- $V^+ : \mathcal{P} \to \wp(S)$ and $V^- : \mathcal{P} \to \wp(S)$ are interpretation functions, such that:
  - if $s, t \in V(p)^+$, then $s \sqcup t \in V(p)^+$ and
  - if $s, t \in V(p)^-$, then $s \sqcup t \in V(p)^-$.

Exact truth and falsemaking

For $\mathbf{M} = \langle S, \sqsubseteq, V^+, V^- \rangle$ a model:

$$
\begin{array}{lll}
\mathbf{M}, s \vDash p & \text{iff} & s \in V(p)^+ \\
\mathbf{M}, s \dashv p & \text{iff} & s \in V(p)^- \\
\mathbf{M}, s \vDash \neg\varphi & \text{iff} & \mathbf{M}, s \dashv \varphi \\
\mathbf{M}, s \dashv \neg\varphi & \text{iff} & \mathbf{M}, s \vDash \varphi \\
\mathbf{M}, s \vDash \varphi \wedge \psi & \text{iff} & \exists t, u \in S(s = t \sqcup u,\ \mathbf{M}, t \vDash \varphi,\ \text{and } \mathbf{M}, u \vDash \psi) \\
\mathbf{M}, s \dashv \varphi \wedge \psi & \text{iff} & \mathbf{M}, s \dashv \varphi,\ \mathbf{M}, s \dashv \psi,\ \text{or } \mathbf{M}, s \dashv \varphi \vee \psi \\
\mathbf{M}, s \vDash \varphi \vee \psi & \text{iff} & \mathbf{M}, s \vDash \varphi,\ \mathbf{M}, s \vDash \psi,\ \text{or } \mathbf{M}, s \vDash \varphi \wedge \psi \\
\mathbf{M}, s \dashv \varphi \vee \psi & \text{iff} & \exists t, u \in S(s = t \sqcup u,\ \mathbf{M}, t \dashv \varphi,\ \text{and } \mathbf{M}, u \dashv \psi)
\end{array}
$$

$$
\begin{array}{lll}
\mathbf{M}, s \vDash \Gamma & \text{iff} & \mathbf{M}, s \vDash \psi, \text{ for all } \psi \in \Gamma
\end{array}
$$

Exact entailment and equivalence

- $\Gamma \vDash \varphi$ iff for all models $\mathbf{M} = \langle S, \sqsubseteq, V^{+}, V^{-} \rangle$ and $s \in S$, if $\mathbf{M}, s \vDash \Gamma$, then $\mathbf{M}, s \vDash \varphi$;

- $\varphi \dashv\vDash \psi$ iff $\varphi \vDash \psi$ and $\psi \vDash \varphi$.

Laws

$$\varphi \vDash \varphi \qquad \text{(reflexivity)}$$

$$\text{if } \Gamma \vDash \varphi, \text{ then } \Gamma, \psi \vDash \varphi \qquad \text{(monotonicity)}$$

$$\text{if } \Gamma \vDash \psi \text{ and } \psi, \Delta \vDash \varphi, \text{ then } \Gamma, \Delta \vDash \varphi \qquad \text{(transitivity)}$$

$$\varphi \dashv\vDash \varphi \wedge \varphi \qquad \varphi \dashv\vDash \varphi \vee \varphi \qquad \text{(idempotency)}$$

$$\varphi \wedge \psi \dashv\vDash \psi \wedge \varphi \qquad \varphi \vee \psi \dashv\vDash \psi \vee \varphi \qquad \text{(commutativity)}$$

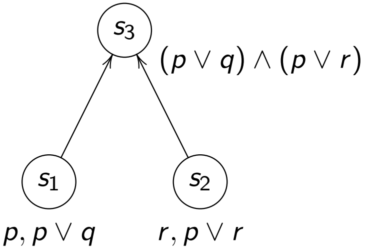
$$\varphi \wedge (\psi \wedge \chi) \dashv\vDash (\varphi \wedge \psi) \wedge \chi \qquad \varphi \vee (\psi \vee \chi) \dashv\vDash (\varphi \vee \psi) \vee \chi \qquad \text{(associativity)}$$

$$\varphi \wedge (\psi \vee \chi) \dashv\vDash (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \qquad (\wedge/\vee\text{-distribution})$$

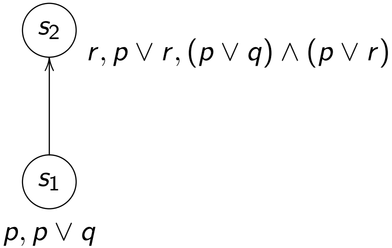
$$\varphi \vee (\psi \wedge \chi) \vDash (\varphi \vee \psi) \wedge (\varphi \vee \chi) \qquad (\vee/\wedge\text{-distribution}^*)$$

$\vee/\wedge$-distribution

(Fine and Jago 2017)



Simplified model



Laws

$$\varphi \vDash \varphi \lor \psi \text{ and } \psi \vDash \varphi \lor \psi \qquad \text{(disjunction introduction)}$$

$$\text{if } \Gamma, \varphi \vDash \chi \text{ and } \Gamma, \psi \vDash \chi, \text{ then } \Gamma, \varphi \lor \psi \vDash \chi \qquad \text{(disjunction elimination)}$$

$$\varphi, \psi \vDash \varphi \land \psi \qquad \text{(conjunction introduction)}$$

$$\varphi, \varphi \land \psi \land \chi \vDash \varphi \land \varphi \qquad \varphi, \varphi \land \psi \land \chi \vDash \varphi \land \chi \qquad \text{(conjunctive convexity)}$$

$$\varphi \mathrel{=\!\mid\!\vDash} \neg\neg\varphi \qquad \text{(double negation)}$$

$$\neg(\varphi \lor \psi) \mathrel{=\!\mid\!\vDash} \neg\varphi \land \neg\psi \qquad \text{($\lor/\land$-de Morgan)}$$

$$\neg(\varphi \land \psi) \mathrel{=\!\mid\!\vDash} \neg\varphi \lor \neg\psi \qquad \text{($\land/\lor$-de Morgan)}$$

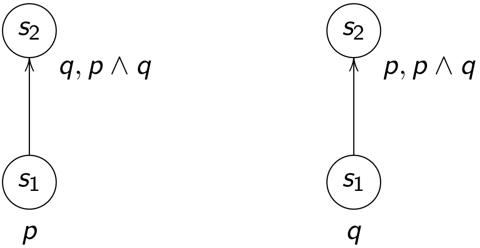
Proofs

- $\varphi, \psi \vDash \varphi \wedge \psi$

  Proof: Suppose that $\mathbf{M}, s \vDash \varphi$ and $\mathbf{M}, s \vDash \psi$. Then, since $s = s \sqcup s$, $\mathbf{M}, s \vDash \varphi \wedge \psi$.

- $p \wedge q \nvDash p$ and $p \wedge q \nvDash q$.

  Proof:



Proofs

- $\varphi, \varphi \wedge \psi \wedge \chi \vDash \varphi \wedge \varphi$ and $\varphi, \varphi \wedge \psi \wedge \chi \vDash \varphi \wedge \chi$

  Proof: Suppose that $\mathbf{M}, s \vDash \varphi$ and $\mathbf{M}, s \vDash \varphi \wedge \psi \wedge \chi$. So there exist t_1, t_2, t_3, such that $s = t_1 \sqcup t_2 \sqcup t_3$ and $\mathbf{M}, t_1 \vDash \varphi$, $\mathbf{M}, t_2 \vDash \psi$, and $\mathbf{M}, t_3 \vDash \chi$.

  Since $s = t_1 \sqcup t_2 \sqcup t_3$, we have $t_2 \sqsubseteq s$. It follows that $s = s \sqcup t_2$. So we have that $\mathbf{M}, s \vDash \varphi$, $\mathbf{M}, t_2 \vDash \psi$, and $s = s \sqcup t_2$, meaning that $\mathbf{M}, s \vDash \varphi \wedge \psi$.

Conjunction laws

- Using set notation, we get:
  - if $\mathbf{M}, s \vDash \bigwedge \Gamma_1, \ldots, \mathbf{M}, s \vDash \bigwedge \Gamma_n$, then $\mathbf{M}, s \vDash \bigwedge \bigcup \{\Gamma_1, \ldots, \Gamma_n\}$;
  - if $\Gamma \subseteq \Delta \subseteq \Sigma$, $\mathbf{M}, s \vDash \bigwedge \Gamma$, and $\mathbf{M}, s \vDash \bigwedge \Sigma$, then $\mathbf{M}, s \vDash \bigwedge \Delta$
- For clarity, this means:
  - $\bigwedge \Gamma_1, \ldots, \bigwedge \Gamma_n \vDash \bigwedge \bigcup \{\Gamma_1, \ldots, \Gamma_n\}$
  - if $\Gamma \subseteq \Delta \subseteq \Sigma$, then $\bigwedge \Gamma, \bigwedge \Sigma \vDash \bigwedge \Delta$.

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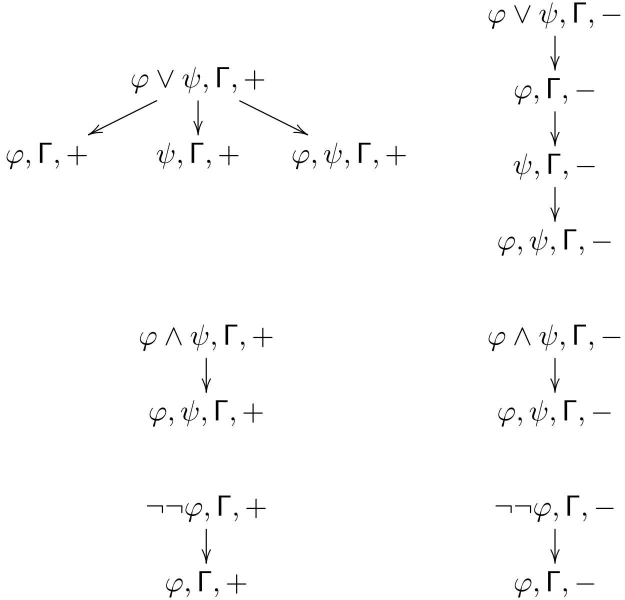
Tableaux

- The entries of our tableaux have the form $\Gamma, +$ and $\Gamma, -$:
  - $\Gamma, +$ means $\mathbf{M}, s \vDash \bigwedge \Gamma$;
  - $\Gamma, -$ means $\mathbf{M}, s \nvDash \bigwedge \Gamma$.
- To test an inference from premises $\Gamma = \{\psi_1, \ldots, \psi_n\}$ to conclusion φ for exact entailment, we start with the *initial list*:

$$\begin{array}{c} \psi_1, + \\ \vdots \\ \psi_n, + \\ \varphi, - \end{array}$$

Rules



Rules

$$\neg(\varphi \vee \psi), \Gamma, + \quad \longrightarrow \quad \neg\varphi, \neg\psi, \Gamma, +$$

$$\neg(\varphi \vee \psi), \Gamma, - \quad \longrightarrow \quad \neg\varphi, \neg\psi, \Gamma, -$$

$$\neg(\varphi \wedge \psi), \Gamma, + \quad \longrightarrow \quad \neg\varphi, \Gamma, + \quad \mid \quad \neg\psi, \Gamma, + \quad \mid \quad \neg\varphi, \neg\psi, \Gamma, +$$

$$\neg(\varphi \wedge \psi), \Gamma, - \quad \longrightarrow \quad \neg\varphi, \Gamma, - \quad \longrightarrow \quad \neg\psi, \Gamma, - \quad \longrightarrow \quad \neg\varphi, \neg\psi, \Gamma, -$$

The idea

- The intuitive idea behind the rules is this:

  Down preservation. If the claim that corresponds to the top node of a rule holds, then at least one of the claims that correspond to the newly generated nodes holds.

  Up preservation. If a claim that corresponds to a node generated by a rule holds, then the claim that corresponds to the top node holds.

- Together, these two properties guarantee soundness and completeness.

The closure rule

- A branch, B, is *closed* iff there are sets Γ and Δ, such that:
  - $\Gamma, + \in B$;
  - $\Delta, - \in B$.
  - $\Gamma \subseteq \Delta \subseteq \bigcup_{\Sigma, + \in B} \Sigma$;

  We mark a closed branch with a $\times$.

- The rationale is just conjunctive convexity:
  - Suppose that $\mathbf{M}, s \vDash \bigwedge \Pi$, for all $\Pi, + \in B$, and $\mathbf{M}, s \nvDash \bigwedge \Pi$, for all $\Pi, - \in B$.
  - Then $\mathbf{M}, s \vDash \bigwedge \Gamma$ and $\mathbf{M} \vDash \bigwedge \bigcup_{\Sigma, + \in B} \Sigma$.
  - Then, since $\Gamma \subseteq \Delta \subseteq \bigcup_{\Sigma, + \in B} \Sigma$, we get $\mathbf{M}, s \vDash \bigwedge \Delta$. $\lightning$

The calculus

- A tableau is *complete* iff every rule that can be applied has been applied.

- $\Gamma \vdash \varphi$ iff there exists a tableau for the inference from Γ to φ in which every branch is closed.

Example

$$p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r)$$

$$p \vee (q \wedge r), +$$
$$(p \vee q) \wedge (p \vee r), -$$

$$\Downarrow$$

$$(p \vee q), (p \vee r), -$$

$$\Downarrow$$

$$p, (p \vee r), -$$

$$\Downarrow$$

$$q, (p \vee r), -$$

$$\Downarrow$$

$$p, q, (p \vee r), -$$

$$\Downarrow$$

$$p, -$$

$$\Downarrow$$

$$p, r, -$$

$$\Downarrow$$

$$p, q, -$$

$$\Downarrow$$

$$q, r-$$

$$\Downarrow$$

$$p, q, r-$$

$p, +$	$q \wedge r, +$	$p, q \wedge r, +$
$\times$	$\Downarrow$	$\Downarrow$
	$q, r, +$	$p, q, r, +$
	$\times$	$\times$

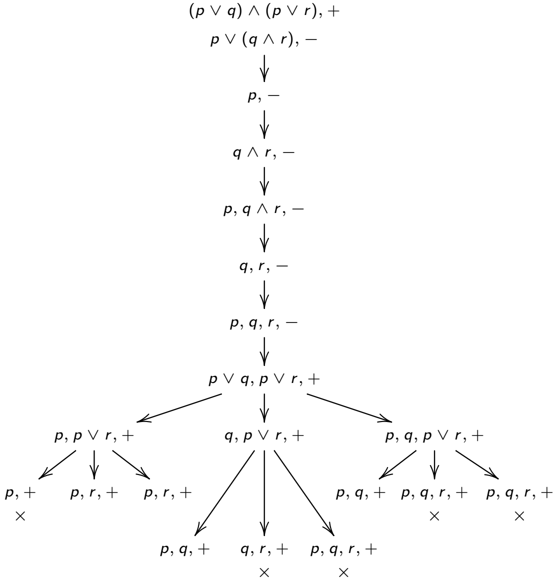
Open branches

- A branch is *open* iff it's not closed.

- $\Gamma \nvdash \varphi$ iff every complete tableau for the inference from Γ to φ has at least one open branch.

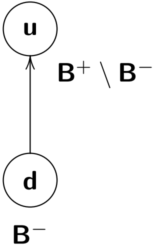
Example

$$(p \vee q) \wedge (p \vee r) \nvdash p \vee (q \wedge r)$$



Countermodels

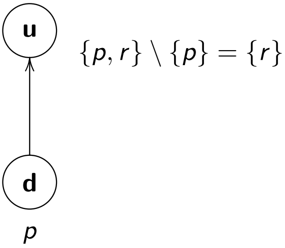
- An open branch of a complete tableau determines a countermodel, $\mathbf{M}_B$, as follows:
  - $\mathbf{B}^- = \bigcup\{\Gamma \subseteq \Lambda : \Gamma, - \in B \text{ and } \exists\Delta(\Delta, + \in B \text{ and } \Gamma \subseteq \Delta)\}$
  - $\mathbf{B}^+ = \bigcup\{\Gamma \subseteq \Lambda : \Gamma, + \in B\}$



- We get:
  - $\mathbf{M}_B, \mathbf{u} \models \bigwedge \Gamma$, whenever $\Gamma, + \in B$;
  - $\mathbf{M}_B, \mathbf{u} \not\models \bigwedge \Gamma$, whenever $\Gamma, - \in B$.

Example

- In the case of $(p \vee q) \wedge (p \vee r) \nvdash p \vee (q \wedge r)$:
  - $\mathbf{B}^- = \{p\}$
  - $\mathbf{B}^+ = \{p, r\}$
  - We get the following countermodel:



- This is just our minimal countermodel from before.

Meta-results

- Soundness and Completeness: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$.
- Decidability: For finite Γ, there is an effective algorithm for determining *whether* $\Gamma \vDash \varphi$.
- Finite Model Property: If $\Gamma \nvDash \varphi$, then there's a countermodel with at most 2 states.

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The theorem

- The canonical model $\mathfrak{M} = \langle S_{\mathfrak{M}}, \sqsubseteq_{\mathfrak{M}}, V^+_{\mathfrak{M}}, V^-_{\mathfrak{M}} \rangle$ is defined by:
  - $S_{\mathfrak{M}} = \wp(\Lambda)$
  - $\sqsubseteq_{\mathfrak{M}} = \subseteq_{\restriction_{\wp(\Lambda)}}$
  - $V^+_{\mathfrak{M}}(p) = \{\{p\}\}$
  - $V^-_{\mathfrak{M}}(p) = \{\{\neg p\}\}$
- $\mathcal{S}(\varphi) = \{\Gamma \subseteq \Lambda : \mathfrak{M}, \Gamma \vDash \varphi\}$
- $\mathcal{S}(\Gamma) = \{\{f(S(\psi)) : \psi \in \Gamma\} \mid f \text{ a choice function}\}$
- Fine-Jago Theorem: $\Gamma \vDash \varphi$ iff for all $X \in \mathcal{S}(\Gamma)$ there is a $y \in \mathcal{S}(\varphi)$ such that for some $x \in X$, $x \subseteq y \subseteq \bigcup X$.

Selection and tableaux

- We can prove the Fine-Jago theorem via the soundness and completeness of our tableaux:
  - For each B, if $\Delta, + \in B$ and $\Delta \subseteq \Lambda$, then there is an $X \in \mathcal{S}(\Gamma)$ with $\Delta \in X$.
  - And if $\Delta, - \in B$ with $\Delta \subseteq \Lambda$, then $\Delta \in \mathcal{S}(\varphi)$.
  - In fact, for each $X \in \mathcal{S}(\Gamma)$ and $\Delta \in X$, there is a B with $\Delta, + \in B$.
  - And for each $\Delta \in \mathcal{S}(\varphi)$, $\Delta, - \in B$ for *all* B.
  - The condition of the theorem is the closure condition of our tableaux.

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- We get a semantic interpretation of the Fine-Jago theorem in terms of the behavior of conjunction.
- What does the law of conjunctive convexity *mean*?
- $\psi_1, \ldots, \psi_n$ jointly weakly ground φ iff $\psi_1 \wedge \ldots \psi_n \vDash \varphi$.
- There are several technical applications of the tableaux (sequent calculus, natural deduction, extensions, modifications, ...).

Thanks!