Truthmakers and Normative Conflicts

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Abstract

By building on work by Kit Fine, we develop a sound and complete truthmaker semantics for Lou Goble's conflict tolerant deontic logic **BDL**.

1 Introduction

A *normative conflict* consists of a number of obligations that cannot jointly be met. Normative conflicts arise, for example, in the context of *moral dilemmas*. Think of Sartre's famous example of the French student [12]: During WW2, the student faces a choice between two possible courses of action: he can either join the French resistance to avenge his brother, who was killed by the invading German forces, or he can take care of his widowed mother, of whom he is the only living relative. The student cannot do both, but, as Sartre argues, the student has an obligation to do both. He faces a normative conflict. A *conflict tolerant deontic logic* (CTDL) is a deontic logic in which normative conflicts are consistent. CTDLs are needed when one, like Sartre, holds that there are real normative conflicts, which cannot be dissolved, e.g. by showing that what seems like an obligation isn't really one. When faced with a normative conflict, one should still be able reason about the implications of ones (conflicting) obligations, to decide what's the best course of action. A conflict *in*tolerant deontic logic, which renders normative conflicts inconsistent, would, of course, not be of much help here: since from a contradiction anything follows, in such a logic one cannot reasonably reason about the conflict at hand. This is *the problem of conflict tolerance.*

How should such a CTDL look like? In a recent survey, Lou Goble sets out to answer this question, and he proposes three desiderata a CTDL should satisfy [11]:

Consistent Conflicts. At least some normative conflicts should to be consistent, i.e. we can have $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$ but $OA_1, \ldots, OA_n \nvdash \bot$ (p. 297).

No Deontic Explosion. Normative conflicts should not result in *deontic explosion*, i.e. we can have $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$ but $OA_1, \ldots, OA_n \nvdash OB$ (p. 298).

Minimal Deontic Laws. Certain minimal laws of deontic logic, which are plausible from considerations independent of any particular view of deontic conflicts, should be validated (p. 302). As examples, Goble explicitly mentions:

(DDS)	$O(A \vee B), O\neg A \vdash OB$	('deontic disjunctive syllogism').
(M)	$O(A \wedge B) \vdash OB$	('monotonicity').
(AGG)	$OA, OB \vdash O(A \wedge B)$	('aggregation').

There is a number of different options to build deontic logics that satisfy the above desiderata.[1] In this paper, we're interested in what Goble calls logics "with limited replacement", i.e. systems in which the rule of substitution can only be applied for a restricted class of statements. In particular, we're interested in Goble's system **BDL**, in which substitution is only allowed for what Goble calls *analytically* equivalent statements [11, p. 315–18].

BDL is a promising candidate for a CTDL since it satisfies all of Goble's desiderata. But as Goble himself points out:

> On the formal front, **BDL** [...] so far lack[s] any semantics or model theory, and it is difficult to see how that might be developed, while respecting the limits necessary to protect their treatment of normative conflicts. [11, p. 318]

In this paper, we develop a sound and complete semantics for **BDL**. By building on Kit Fine's recent truthmaker semantics for analytic equivalence [8], we are able to meet Goble's challenge. We propose new, conflict tolerant semantic clauses for statements of obligation, which are partially inspired by Fine's truthmaker treatment of statements of permission [7]. Based on a notion of admissible states, Kit Fine essentially proposed the following truth-condition for permission statements of the form PA (for "it is permitted that A"):

- PA is true iff every state that is a truthmaker of A is admissible.

We will now use this very notion of admissible states to develop a semantics for obligation. Put intuitively, our truth-condition for obligation statements is this:

- OA is true iff there is no admissible state that is a falsemaker of A.

In what follows, we start by developing a sound and complete semantics for the weaker system $\mathbf{BDL}^-$.[2] As will become clear later in the paper (section 4), we take $\mathbf{BDL}^-$ to be a better CTDL than Goble's preferred logic **BDL**. However, by adding an additional semantic constraint, we'll also be able obtain the validity of (DDS), resulting in a sound and complete semantics for full **BDL**.

The structure of the paper is as follows. In section §2, we discuss the limited replacement approach to CTDLs and introduce Goble's system **BDL**. We show that Goble's proposed system for analytically equivalence is deductively equivalent to Angell's system AC of analytic containment [2]. In the following section, §3, we extend Kit Fine's truthmaker semantics for AC with our conflict tolerant semantic clauses for obligation. In §4, we give a semantic constraint on our models that is equivalent to the validity of (DDS), thereby obtaining a sound and complete semantics for **BDL**. In 5, we prove the main result of our

[1] For an overview of these options, see §4 and §5 of [11].

[2] Which is **BDL** without the rule (DDS).

paper: the soundeness of completeness of our semantics $\mathbf{BDL}^-$ and **BDL**. We conclude with a discussion of (DDS), arguing that it is *not* a plausible principle in a setting that allows for normative conflicts.

Syntax

The syntax in our paper is as follows: Our *base language* $\mathcal{L}$ is a propositional language with the connectives $\neg$ ('negation'), $\wedge$ ('conjunction'), and $\vee$ ('disjunction'), which is defined over a set $\mathcal{A}$ of propositional variables or *atoms*. We use $p, q, r, \ldots$ as meta-variables for atoms. The syntax of $\mathcal{L}$ is given in a concise fashion by the following Backus-Naur-Form:

$$A ::= p \mid \neg A \mid (A \wedge A) \mid (A \vee A).$$

We use $A, B, C, \ldots$ as meta-variables for formulas. We also refer to the formulas of $\mathcal{L}$ as *non-deontic* formulas. The *deontic language* $\mathcal{L}_D$ extends $\mathcal{L}$ with formulas in which the obligation operator O has been applied to non-deontic formulas, i.e.

$$\mathcal{L}_D := \mathcal{L} \cup \{OA : A \in \mathcal{L}\}.$$

The formulas in $\mathcal{L}_D \setminus \mathcal{L}$ we also call *deontic* formulas. Throughout the paper, the usual notational conventions about formulas apply: outermost brackets may be omitted; $\neg$ binds stronger than $\wedge$, which in turn binds stronger than $\vee$; etc. $A \to B$ is defined as $\neg A \vee B$ and $A \leftrightarrow B$ is defined as $(A \to B) \wedge (B \to A)$.

2 CTDLs with Limited Replacement

The *rule of replacement* (RE) says that we can infer OB from OA given that $\vdash A \leftrightarrow B$. Now the problem is this: any CTDL that contains the rule replacement and takes classical logic as its background logic either violates **No Deontic Explosion** or **Minimal Deontic Laws**. The argument is relatively straightforward. Suppose that a CTDL contains (RE) and classical logic as its background logic. Now according to **Minimal Deontic Laws**, it also contains (M). Then using (RE) and (M), we can derive the *rule of monotonicity* (RM), which says that we can infer OB from OA, if $\vdash A \to B$.[3] Since our background logic is classical, (RM) immediately results in deontic explosion: it follows from $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$ that $\vdash A_1 \wedge \ldots \wedge A_n \to B$; but then we can infer OB from $OA_1, \ldots, OA_n$ using (RM), in direct contradiction to **No Deontic Explosion**. Hence no CTDL with classical logic as its background logic can satisfy both **No Deontic Explosion** and **Minimal Deontic Laws**.

In fact, (RM) is *equivalent* to (RE) and (M) in the sense that any deontic logic that has (RM) has (RE) and (M) and vice versa.[4] In other words, in the light of the rule (RE) and classical logic, we cannot distinguish between the problematic rule (RM) and the desired axiom (M). This observation motivates

[3]Suppose $\vdash A \to B$. By classical logic we get $\vdash A \leftrightarrow A \wedge B$. Now suppose that we have a derivation of OA. Using (RM), we can infer $O(A \wedge B)$. Using (M) and conjunction elimination we can derive OB.

[4]Above we've shown how to derive (RM) from (RE) and (M) using classical logic. For the converse direction, simply note that (RE) follows by applying (RM) "in both directions" and (M) follows from (RM) using that fact that $A \wedge B \vdash A$ and $A \wedge B \vdash B$.

restricting the replacement rule, giving us the class of CTDLs with *limited replacement* [11, §5.4].[5] The idea is that we no longer sanction the inference from OA to OB only on the basis of $\vdash A \leftrightarrow B$, but rather demand a stronger form of equivalence to hold between A and B as a side-condition for replacement. The question is: What does a plausible such notion of equivalence look like?

Goble proposes an interesting CTDL with limited replacement, which instead of (classical) logical equivalence, uses the stronger condition of 'analytic equivalence' as the condition for replacement in deontic contexts. Goble formalizes this notion using the binary operator $\Leftrightarrow_A$, which operates on non-deontic formulas. More specifically, an *analytic equivalence claim* is a statement of the form $A \Leftrightarrow_A B$, where $A, B \in \mathcal{L}$. Goble proposes the following axiomatization for analytic equivalences [11, p. 316]:[6]

Axioms:

$A \Leftrightarrow_A A$	$A \Leftrightarrow_A \neg\neg A$
$A \Leftrightarrow_A (A \wedge A)$	$A \Leftrightarrow_A (A \vee A)$
$(A \wedge B) \Leftrightarrow_A (B \wedge A)$	$(A \vee B) \Leftrightarrow_A (B \vee A)$
$(A \wedge (B \wedge C)) \Leftrightarrow_A ((A \wedge B) \wedge C)$	$(A \vee (B \vee C)) \Leftrightarrow_A ((A \vee B) \vee C)$
$(A \wedge (B \vee C)) \Leftrightarrow_A ((A \wedge B) \vee (A \wedge C))$	$(A \vee (B \wedge C)) \Leftrightarrow_A ((A \vee B) \wedge (A \vee C))$
$(\neg A \wedge \neg B) \Leftrightarrow_A \neg(A \vee B)$	$(\neg A \vee \neg B) \Leftrightarrow_A \neg(A \wedge B)$

Rules:

(R1) $A \Leftrightarrow_A B / B \Leftrightarrow_A A$	(R2) $A \Leftrightarrow_A B, B \Leftrightarrow_A C / A \Leftrightarrow_A C$
(R3) $A \Leftrightarrow_A B / (A \wedge C) \Leftrightarrow_A (B \wedge C)$	(R4) $A \Leftrightarrow_A B / (A \vee C) \Leftrightarrow_A (B \vee C)$
(R5) $A \Leftrightarrow_A B / \neg A \Leftrightarrow_A \neg B$	

It turns out that Goble's system is deductively equivalent to the well-understood system AC of analytic containment, which is due to RB Angell [1, 2]. In [8], Fine gives an axiomatization of AC that is almost identical to Goble's system, except that Fine's system neither has the identity axiom $A \Leftrightarrow_A A$ nor the negation replacement rule $A \Leftrightarrow_A B / \neg A \Leftrightarrow \neg B$ (R5). It's easily shown that we can derive the identity axiom from the other axioms of Goble's system, which makes the axiom redundant:

1. $A \Leftrightarrow_A \neg\neg A$ (Axiom)
2. $\neg\neg A \Leftrightarrow_A A$ (from 1. using R1)
3. $A \Leftrightarrow_A A$ (from 1. and 2. using R2)

Moreover, Fine shows that the negation replacement rule is admissible in the system (Theorem 2, [8, p. 203]), establishing that Goble's system indeed is just AC.

Proposition 2.1. *$\vdash A \Leftrightarrow_A B$ in Goble's system iff $\vdash_{AC} A \Leftrightarrow_A B$.*

[5] Another possible response to the problem sketched above is, of course, to abandon classical logic as the background logic for our CTDL. For a discussion of this approach, see [11, p. 321–26].

[6] Goble includes the axiom $(A \to B) \Leftrightarrow_A (\neg A \vee B)$, which we don't include here since we define $A \to B$ as $\neg A \vee B$.

In a sense, we wish to argue, this observation vindicates Goble's choice of system: it turns out that Goble's system coincides with a well-known system for a non-classical notion of equivalence, which has been independently studied by philosophers and logicians [5, 6, 8]. Moreover, there is a semantics for AC, which, as we'll show in the next section, can be extended in a natural fashion to account for deontic formulas.

Goble now defines a CTDL in which replacement is restricted to analytic equivalences. The system **BDL** of 'basic deontic logic' is formulated in $\mathcal{L}_D$ and consists of classical propositional logic, plus the rules (DDS), (M), and (AGG), as well as the replacement rule (RBE), which allows us to infer OB from OA given that $\vdash_{AC} A \Leftrightarrow_A B$ [11, p. 314]. Note that AC plays the role of a "background system" here: derivability in AC of a certain analytic equivalence is a side-condition for the rule (RBE) in the system **BDL**, but AC itself is not part of **BDL**.

BDL and BDL$^-$

The system **BDL** has the following axioms and rules:

Axioms

1. all substitution instances of classical tautologies over the language $\mathcal{L}_D$
2. $O(A \wedge B) \to (OA \wedge OB)$ (M)
3. $(OA \wedge OB) \to O(A \wedge B)$ (AGG)

Rules

$$\frac{A \to B \quad A}{B}\ (MP) \qquad \frac{O(A \vee B) \quad O\neg A}{OB}\ (DDS)$$

$$\frac{OA}{OB}\ {\scriptstyle \vdash_{AC} A \Leftrightarrow_A B}\ (RBE)$$

The system **BDL$^-$** is **BDL** without the rule (DDS).

We denote derivability in **BDL$^-$** by $\vdash_{\mathbf{BDL}^-}$ and derivability in **BDL** by $\vdash_{\mathbf{BDL}}$. If it's clear from the context which system we're talking about, we may omit the subscript.

Before we start with semantics, let us briefly point out a few facts that'll turn out to be useful later in the paper.

First, note that the following rule

$$\frac{OA}{OB}\ {\scriptstyle \vdash_{AC} A \wedge B \Leftrightarrow_A A}\ (KBE)$$

is derivable in both **BDL$^-$** and **BDL** using (RBE) and (M). We call it (KBE) for its resemblance to the K rule of deontic modal logic.

Second, note that using the rule (KBE), we can show that

$$\vdash O(A \wedge B) \to O(A \vee B)$$

by observing that $\vdash_{AC} (A \wedge B) \wedge (A \vee B) \Leftrightarrow_A (A \wedge B)$.

And finally, note that:

Lemma 2.2. $\Phi \vdash_{\mathbf{BDL}} A \Leftrightarrow \Phi \cup \{O(A \vee B) \wedge O\neg A \rightarrow OB : A, B \in \mathcal{L}\} \vdash_{\mathbf{BDL}^-} A$

3 Truthmaker Semantics for BDL$^-$

Fine formulates his semantics for AC in a modified version of Bas van Fraassen's truthmaker semantics [10]. Van Fraassen originally used his semantics to give a characterization of what's effectively the 4-valued logic of First Degree Entailment, and in his paper, Fine shows how the semantics can be extended to AC [8].

Here we present slightly different version of the semantics, which as far as we can see, was first suggested in passing by Stephen Yablo [13, p. 57]. The semantics is developed against a background theory of fine-grained *states of affairs*. These states are taken to be primitive entities of the semantics, not reducible to the possible worlds where they obtain or the like. Formally speaking, a state is a sets of pairs of atoms and truth-values:

Definition 3.1 (State)**.** *σ is a state iff $\sigma \subseteq \mathcal{A} \times \{0, 1\}$.*

States can be "negated" in a natural way:

Definition 3.2 (Negation of a state)**.** *Let σ be a state. We write $\overline{\sigma}$ for the negation of σ, and define $\overline{\sigma} := \{(p_i, 1 - x) : (p_i, x) \in \sigma\}$.*

Philosophically, we can think of our states as *Ersatz*-states in the same way that Carnapian state descriptions are *Ersatz*-worlds [4]. We read the state $\{(p_1, x_1), (p_2, x_2), \ldots\}$ as the state of p_1 having the truth-value x_1, p_2 having the truth-value x_2, $\ldots$.

Note that this definition allows for incomplete and inconsistent states. A state σ is said to be *incomplete* iff there is a $p \in \mathcal{A}$ such that neither $(p, 1) \in \sigma$ nor $(p, 0) \in \sigma$. And σ is said to be *inconsistent* iff both $(p, 1) \in \sigma$ and $(p, 0) \in \sigma$ for some $p \in \mathcal{A}$. Intuitively, an incomplete state is one that fails to settle a certain subject matter, and an inconsistent state is one that is over-determined with respect to some subject matter.

We can think of classical valuations as special kind of states in the following way:

Definition 3.3 (Classical Valuation)**.** *ω is a classical valuation iff ω is a state and for every $p \in \mathcal{A}$, either $(p, 1) \in \omega$ or $(p, 0) \in \omega$.*

In fact, mathematically speaking, that's just what valuations are: functions from $\mathcal{A}$ to $\{0, 1\}$. Philosophically speaking, classical valuations are just *Ersatz*-worlds: they are technical stand-ins for ways the world can be.

The idea behind truthmaker semantics is that states are those things in the world that make statements true or false. Fine gives us the following informal characterization of truthmaking: he says that σ is a truthmaker of A just in case (i) σ necessitates the truth of A and (ii) σ is wholly relevant to the truth of A [9, p. 559].[7] There is also an analogous *false*making relation, which holds between

[7] More precisely, this is what Fine calls "exact" truthmaking to distinguish it from truthmaking in other senses. Unless further specified, whenever we speak of the truthmakers of a statement, this is what we mean.

a state σ and a statement A just in case (i') σ necessitates the falsehood of A and (ii') σ is wholly relevant to the falsehood of A.

In our present setting, there will be precisely one state that necessitates the truth of an atom p in a wholly relevant way, which is just the state $\{(p,1)\}$. And there is exactly one state that necessitates the falsehood of p in a wholly relevant way, which is $\{(p,0)\}$. If we start from this and take Fine's recursive clauses for the truthmaking and falsemaking relation, we end up with the following definition:

Definition 3.4 (Truthmakers, Falsemakers)**.** *For all $A \in \mathcal{L}$, the set $[A]^+$ of truthmakers of A and the set $[A]^-$ of falsemakers of A is defined by simultaneous recursion as follows:*

i) a) $[p]^+ = \{\{(p,1)\}\}$
   b) $[p]^- = \{\{(p,0)\}\}$

ii) a) $[\neg A]^+ = [A]^-$
   b) $[\neg A]^- = [A]^+$

iii) a) $[A \wedge B]^+ = \{\sigma \cup \tau : \sigma \in [A]^+, \tau \in [B]^+\}$
   b) $[A \wedge B]^- = [A]^- \cup [B]^- \cup [A \vee B]^-$

iv) a) $[A \vee B]^+ = [A]^+ \cup [B]^+ \cup [A \wedge B]^+$
   b) $[A \vee B]^- = \{\sigma \cup \tau : \sigma \in [A]^-, \tau \in [B]^-\}$

One way of thinking about what's going on here is that we're keeping track of the precise truth and falsity conditions of a given statement: the members of $[A]^+$ are the exact conditions that need to be satisfied by a valuation for A to be true under the valuation, and the members of $[A]^-$ are the exact conditions for A's falsehood.

Fine [8] discusses various notions of semantic content that can be developed in the truthmaker setting. To get a semantics for AC, we need the notion of *replete* content, which is defined in terms of convexity:

Definition 3.5 (Convex set)**.** *A set Σ of states is said to be* convex *if and only if for all states σ, τ, δ, if $\sigma, \delta \in \Sigma$ and $\sigma \subseteq \tau \subseteq \delta$, then $\tau \in \Sigma$.*

Every set of states can canonically be transformed into a convex set, by filling in the missing pieces:

Definition 3.6 (Convex closure)**.** *We define the convex closure, $conv(\Sigma)$, of a set of states Σ as the smallest Θ such that $\Sigma \subseteq \Theta$ and Θ is convex. For the convex closure $conv([A]^+)$ we also write $[\![A]\!]^+$, and for $conv([A]^-)$ we write $[\![A]\!]^-$.*

It's easily checked that this is indeed well-defined.

Following Fine, we call $[\![A]\!]^+$ the set of *replete truthmakers* of A (and $[\![A]\!]^-$ *replete falsemakers* of A). To illustrate the difference between exact and replete truthmakers, it's maybe helpful to look at an example. Take the formula $(p \wedge q \wedge r) \vee p$. This formula has two exact truthmakers: the state that makes the left disjunct (i.e. $(p \wedge q \wedge r)$) true, and the state that makes the right disjunct (i.e. p) true, and this is why we have $[(p \wedge q \wedge r) \vee p]^+ =$

$\{\{(p,1),(q,1),(r,1)\},\{(p,1)\}\}$. The set of replete truthmakers for the same formula now contains additional elements, in our example $[\![(p \wedge q \wedge r) \vee p]\!]^+ = \{\{(p,1),(q,1),(r,1)\},\{(p,1),(q,1)\},\{(p,1),(r,1)\},\{(p,1)\}\}$. Intuitively speaking, the set of replete truthmakers (falsemakers) also contains all those states, that make the formula true (false) but that are also contained in the statement's subject matter.

It can be shown that Goble's axioms and rules for analytic equivalence exactly describe identity of replete truthmakers (or falsemakers, for that matter):

Theorem 3.7 (Replete Truthmakers and Analytic Equivalence). *For all $A, B \in \mathcal{L}$, the following three statements are equivalent:*

- $\vdash_{AC} A \Leftrightarrow_A B$
- $[\![A]\!]^+ = [\![B]\!]^+$
- $[\![A]\!]^- = [\![B]\!]^-$

Proof. By proposition 2.1 and Kit Fine's proof in [8]. □

So in this sense, replete truthmakers are the semantic foundation for analytic equivalence. Besides being an interesting observation in itself, this observation is a partial solution to Goble's challenge to find an intuitive semantical framework for **BDL**, because it gives us a semantic framework to interpret the non-deontic part of the language. To interpret obligations we add one final ingredient to our base model: a set Ok, which contains all those states that are normatively admissible. So this leads us to the following definition of a base model:

Definition 3.8 (Base Model). *A model is a tuple $\mathcal{M} = (\omega, Ok)$, where ω is a classical valuation over $\mathcal{L}$ and $Ok \subseteq \wp(\mathcal{A} \times \{0,1\})$.*

So Ok is a set of states, i.e. a set of sets of ordered pairs atoms and truth-values. To give you an idea of how we interpret these Ok sets, consider the following example. Assume that $Ok = \{\{(p,1),(q,0)\},\{(p,1),(s,1)\}\}$. This Ok set renders two states admissible: the state in which p is true and q is false (e.g. you drink and you don't drive), and the state in which p is true and t is true (e.g. you drink and you take a taxi).

So far there are no conditions on the set Ok of admissible states. A consequence of that is that a complex state $\{(p,1),(s,1)\}$ can be in Ok, without any of its substates, e.g. $\{(p,1)\}$, being in Ok. This enables us to express that two things might be admissible only in combination with one another, while not being admissible in isolation.[8]

To get a semantics for $\mathbf{BDL}^-$ and **BDL**, however, we need Ok so satisfy additional conditions. To validate the axioms of $\mathbf{BDL}^-$, we need to assume that if two states are inadmissible, then any state "in between" will also be inadmissible, and we need to assume that if the combination of two states is admissible, then at least one of the two states is admissible:

Definition 3.9 (Reverse Convexity). *A set of states Σ is* reverse convex *iff the complement Σ^C with respect to the set of all states is convex, i.e. for all σ, τ, π: if $\sigma \notin \Sigma$ and $\tau \notin \Sigma$ and $\sigma \subseteq \pi \subseteq \tau$, then $\pi \notin \Sigma$.*

[8]We will come back to this closure condition for Ok later in section 4 when we talk about (DDS) and a semantics for the stronger logic **BDL**.

Definition 3.10 (Reverse Closure)**.** *A set of states* Σ *is* reverse closed *iff* Σ^C *is closed under union, i.e. for any two states* $\sigma, \tau \notin \Sigma$*, we have* $\sigma \cup \tau \notin \Sigma$.

To illustrate what the reverse convexity condition says, consider an example in which it is violated. For instance, take the *Ok* set that only renders the state admissible which makes p and r true, $Ok = \{\{(p, 1), (r, 1)\}\}$. As a consequence, we have $\{(p, 1), (r, 1), (s, 1)\} \notin Ok$ and $\{(p, 1)\} \notin Ok$. So neither is the state admissible which makes p, r and s true, nor is the state admissible which makes p true. Intuitively speaking, this means that there is an admissible state such that neither a stronger nor a weaker state is admissible. Reverse convexity excludes this situation, i.e. it excludes *Ok* sets like the one in this examples.[9]

These two conditions now finally give us the notion of a $\mathbf{BDL}^-$ model:

Definition 3.11 ($\mathbf{BDL}^-$ model)**.** *A* $\mathbf{BDL}^-$ *model is model* $\mathcal{M} = (\omega, Ok)$ *such that Ok is reverse convex and reverse closed.*

Since the base logic of $\mathbf{BDL}^-$ is classical, the notion of a model is based on a classical valuation, with the usual truth-conditions for non-deontic formulas. For obligations, we use the following idea: what it means for A to be obligatory is that *no replete falsemaker of* A is admissible. Put in terms of truthmakers, this means that A is obligatory iff *no replete truthmaker of* $\neg A$ is admissible.[10] Putting all these conditions together results in the following truth-conditions for all formulas of the deontic language $\mathcal{L}_D$:

Definition 3.12 (Truth)**.** *Let* $\mathbf{M} = (w, Ok)$ *be a* $\mathbf{BDL}^-$ *model. The truth-conditions for* $p \in \mathcal{A}$ *and for non-deontic formulas are classical:*

(i) $\mathcal{M} \vDash p$ *iff* $\omega(p) = 1$

(ii) $\mathcal{M} \vDash \neg A$ *iff* $\mathcal{M} \nvDash A$

(iii) $\mathcal{M} \vDash A \wedge B$ *iff* $\mathcal{M} \vDash A$ *and* $\mathcal{M} \vDash B$

(iv) $\mathcal{M} \vDash A \vee B$ *iff* $\mathcal{M} \vDash A$ *or* $\mathcal{M} \vDash B$

(v) $\mathcal{M} \vDash OA$ *iff* $[\![A]\!]^- \cap Ok = \emptyset$

Validity and logical consequence are defined as usual:

Definition 3.13 (Validity)**.** *For all* $A \in \mathcal{L}_D$,

$$\vDash A \textit{ iff for all } \mathbf{BDL}^- \textit{ models } \mathcal{M}\textit{: } \mathcal{M} \vDash A.$$

Definition 3.14 (Logical Consequence)**.** *For all* $\Phi \subseteq \mathcal{L}_D$ *and* $A \in \mathcal{L}_D$,

$$\Phi \vDash A \textit{ iff for all } \mathbf{BDL}^- \textit{ models } \mathcal{M}, \textit{ if } \mathcal{M} \vDash \Phi, \textit{ then } \mathcal{M} \vDash A.$$

To illustrate how our semantics works, we now consider two concrete models. The first one shows that, although the semantics validates (M), obligations are not generally closed under logical consequence. In particular, we show that that weakening in the form of $OA \to O(A \vee B)$ is not valid. The second one shows that normative conflicts are satisfiable.

[9] Note that both conditions (reverse convexity and reverse closure) really concern the complement of a set of states. Why this is so, will become apparent soon.

[10] Note that we have $[\![A]\!]^- = [\![\neg A]\!]^+$.

(i) Let $\mathcal{M} = (w, Ok)$ with

$$Ok = \{\{(q,0)\}, \{(p,0),(q,0)\}\} \cup \{\sigma : \{(p,0),(q,0)\} \subseteq \sigma\}.$$

Since $[\![p]\!]^- = \{\{(p,0)\}\}$, we have $[\![p]\!]^- \cap Ok = \varnothing$ and by the truth-condition for obligation formulas $\mathcal{M} \vDash Op$. However, in light of the fact that $[\![p \vee q]\!]^- = \{\{(p,0),(q,0)\}\}$ we get $[\![p \vee q]\!]^- \cap Ok \neq \varnothing$, i.e. $\mathcal{M} \nvDash O(p \vee q)$. Hence, we get $\mathcal{M} \nvDash Op \rightarrow O(p \vee q)$.

(ii) Let $\mathcal{M} = (w, Ok)$ with $Ok = \{\{(p_i, 1)\} : i \geq 2 \text{ and } p_i \in \mathcal{A}\}$. So Ok renders a state admissible that makes p_2 true, one that makes p_3 true, and so on. According to the truth-condition for obligations, this makes $\neg p_2$, $\neg p_3$, etc. not obligatory. So we have: $\mathcal{M} \vDash \neg O \neg p_i$, for all $i \geq 2$. What about p_1, though? There is neither a truthmaker nor a falsemaker of p_1 in Ok, i.e. $[\![p_1]\!]^- \cap Ok = \varnothing$ and $[\![\neg p_1]\!]^- \cap Ok = \varnothing$.[11] And this means that $\mathcal{M} \vDash Op_1$ and $\mathcal{M} \vDash O\neg p_1$, i.e. $\mathcal{M}$ satisfies a normative conflict. Despite the fact that normative conflicts are satisfiable, it is also easily shown that $\mathbf{BDL}^-$ base logic is classical:

Lemma 3.15 (Classicality)**.** *For all $A \in \mathcal{L}_D$ and all $\mathbf{BDL}^-$ models, either $\mathcal{M} \vDash A$ or $\mathcal{M} \nvDash A$ and never both.*

This lemma, and the fact that in the example above we have $\mathcal{M} \vDash \neg O \neg p_i$, for all $i \geq 2$, also shows that our truth-conditions allow for a logic that satisfies Goble's desiderata of **No Deontic Explosion** and **Consistent Conflicts**.

4 On Deontic Disjunctive Syllogism and BDL

In the semantics for $\mathbf{BDL}^-$, (DDS) is not sound, and it's easy to construct a countermodel: let $\mathcal{M} = (\omega, Ok)$ be such that

- ω is arbitrary, and
- $Ok = \{\{(q,0)\}\}$.

Note that Ok is reverse convex and reverse closed. We have $\mathcal{M} \vDash O\neg p$, $\mathcal{M} \vDash O(p \vee q)$ but $\mathcal{M} \nvDash Oq$, since $[\![q]\!]^- = \{\{(q,0)\}\} \cap Ok \neq \emptyset$. Hence (DDS) is not a sound rule for our semantics.

To obtain a sound and complete semantics for **BDL** we need a condition on $\mathbf{BDL}^-$ models which guarantees that (DDS) preserves truth in the models that satisfy the condition. This is what we set out to do in this section.

First, we need some further auxiliary concepts. Just like we can negate states, we can also negate sets of states, i.e. *contents*:

Definition 4.1 (Negation of a content)**.** *Let $\sigma_1, \ldots \sigma_n$ be states. We define:*

$$\overline{\{\sigma_1, \ldots \sigma_n\}} = \{\overline{\{f(\sigma_1), \ldots, f(\sigma_n)\}} : f : \{\sigma_1, \ldots, \sigma_n\} \rightarrow \bigcup_{i=1}^{n} \sigma_i \text{ is a choice function}\}$$

For simplicity, let's also introduce the notion of *fusions* of contents:

Definition 4.2 (Fusion of Contents)**.** *We call $\Sigma \circ \Lambda$ the fusion of two sets of sets Σ and Λ, and define $\Sigma \circ \Lambda = \{\sigma \cup \tau : \sigma \in \Sigma \text{ and } \tau \in \Lambda\}$.*

[11] Note that for all $p \in \mathcal{A}$ we have $[p]^+ = [\![p]\!]^+$ and $[p]^- = [\![p]\!]^-$.

We'll use the following corollary every now and then later in the paper:

Corollary 4.3. *For all $A \in \mathcal{L}$, we have $[A]^+ \circ [A]^+ = [A]^+$ and $[A]^- \circ [A]^- = [A]^-$.*

Lemma 4.4. *For all $A \in \mathcal{L}$,*

- $\overline{[A]^+} = [\neg A]^+$
- $\overline{[A]^-} = [\neg A]^-$

Proof. By induction on complexity. For the base case, note that:

- $[p]^+ = [\neg p]^- = \{(p,1)\}$ and $\overline{\{(p,1)\}} = \{(p,0)\}$
- $[\neg p]^+ = [p]^- = \{(p,0)\}$ and $\overline{\{(p,0)\}} = \{(p,1)\}$

For $\neg A$, note that

- $\overline{[\neg A]^+} = \overline{[A]^-} \overset{IH}{=} [\neg A]^- = [A]^+ \overset{!}{=} [\neg\neg A]^+$
- $\overline{[\neg A]^-} = \overline{[A]^+} \overset{IH}{=} [\neg A]^- = [A]^- \overset{!}{=} [\neg\neg A]^-$

We only consider the case for $A \vee B$, leaving the $A \wedge$ for the interested reader:

- $\overline{[A \vee B]^+} = \overline{[A]^+ \cup [B]^+ \cup [A]^+ \circ [B]^+} \overset{!}{=} \overline{[A]^+} \circ \overline{[B]^+} \circ \overline{[A]^+ \cup [B]^+} \overset{!}{=} \overline{[A]^+} \circ \overline{[B]^+} \circ \overline{[A]^+} \circ \overline{[B]^+} = \overline{[A]^+} \circ \overline{[B]^+} \overset{IH}{=} [\neg A]^+ \circ [\neg B]^- = \ldots = [\neg(A \vee B)]^+$
- $\overline{[A \vee B]^-} = \overline{[A]^- \circ [B]^-} \overset{!}{=} \overline{[A]^- \circ [B]^- \circ ([A]^- \cup [B]^-)} \overset{!}{=} \overline{[A]^-} \cup \overline{[B]^-} \cup \overline{[A]^-} \circ \overline{[B]^-} \overset{IH}{=} [\neg A]^- \cup [\neg B]^- \cup [\neg A]^- \circ [\neg B]^- = \ldots = [\neg(A \vee B)]^-$

□

It immediately follows:

Corollary 4.5.

1. $[\![\neg A]\!]^+ = conv(\overline{[A]^+})$
2. $[\![\neg A]\!]^- = conv(\overline{[A]^-})$

Definition 4.6 (Trace formulas)**.** *For all states σ, we set*

$$tr(\sigma) = \bigwedge(\{p : (p,1) \in \sigma\} \cup \{\neg p : (p,0) \in \sigma\})$$

And for all sets of states Σ, we set:

$$tr(\Sigma) = \bigvee\{tr(\sigma) : \sigma \in \Sigma.\}$$

Lemma 4.7. *For all states σ and all sets of states Σ,*

1. $[tr(\sigma)]^+ = \{\sigma\}$
2. $[tr(\Sigma)]^+ = \Sigma$

The following theorem illustrates that (DDS)'s validity corresponds to a certain closure property of *Ok* sets:

Theorem 4.8. *Let $\mathfrak{M}$ be a class of* **BDL**$^-$*-models. Then, the following are equivalent:*

1. *for all $A, B, C \in \mathcal{L}$ and for all $\mathcal{M} \in \mathfrak{M}$, $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$,*

2. *for all $\mathcal{M} = (\omega, Ok) \in \mathfrak{M}$, for all sets of states Σ, Δ, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$, then $conv(\overline{\Delta}) \cap Ok = \emptyset$.*

Proof. 1. $\Rightarrow$ 2.. Assume that for all $A, B, C \in \mathcal{L}$ and for all $\mathcal{M} \in \mathfrak{M}$, $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$. For proof by contradiction, assume that there is an $\mathcal{M}$ and Σ, Δ with $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$ but $conv(\overline{\Delta}) \cap Ok \neq \emptyset$. Consider $O(tr(\Sigma) \vee tr(\Delta))$, $O\neg tr(\Sigma)$, and $Otr(\Delta)$. Using Lemmas 4.4 and 4.7, we get $[tr(\Sigma) \vee tr(\Delta)]^- = [tr(\Sigma)]^- \circ [tr(\Delta)]^- = \overline{\Sigma} \circ \overline{\Delta}$. So $[\![tr(\Sigma) \vee tr(\Delta)]\!]^+ = conv(\overline{\Sigma} \circ \overline{\Delta})$. But then, we get $\mathcal{M} \vDash O(tr(\Sigma) \vee tr(\Delta))$, since by assumption $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$. In a similar fashion, we can determine that $\mathcal{M} \vDash O\neg tr(\Sigma)$ and $\mathcal{M} \nvDash Otr(\Delta)$, which is in contradiction to the assumption that $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$, for all A, B. Hence for all $\mathcal{M} = (\omega, Ok) \in \mathfrak{M}$, for all sets of states Σ, Δ, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\overline{\Sigma}) \cap Ok = \emptyset$, then $conv(\Delta) \cap Ok = \emptyset$.

2. $\Rightarrow$ 1. Conversely, assume that for all $\mathcal{M} = (\omega, Ok) \in \mathfrak{M}$, for all sets of states Σ, Δ, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\overline{\Sigma}) \cap Ok = \emptyset$, then $conv(\Delta) \cap Ok = \emptyset$. This means that for any formulas A, B, if

$$\underbrace{conv(\overline{[A]^+} \circ \overline{[B]^+})}_{=[\![A \vee B]\!]^-} \cap Ok = \emptyset$$

and

$$\underbrace{con([\neg A]^+)}_{=[\![A]\!]^-} \cap Ok = \emptyset$$

then

$$\underbrace{con(\overline{[B]^+})}_{=[\![B]\!]^-} \cap Ok \neq \emptyset$$

Which is just to say that if $\mathcal{M} \vDash O(A \vee B)$ and $\mathcal{M} \vDash O\neg A$, then $\mathcal{M} \vDash OB$. In other words, $\mathcal{M} \vDash O(A \vee B) \wedge O\neg A \rightarrow OB$.

□

Because of this fact we can define **BDL** models as follows:

Definition 4.9 (**BDL** model)**.** *A **BDL** model is a **BDL**$^-$ model $\mathcal{M} = (\omega, Ok)$ in which Ok also satisfies:*

> *For all sets of states Σ, Δ, if $conv(\overline{\Sigma} \circ \overline{\Delta}) \cap Ok = \emptyset$ and $conv(\Sigma) \cap Ok = \emptyset$, then $conv(\overline{\Delta}) \cap Ok = \emptyset$.*

Validity and logical consequence for **BDL** are defined as usual:

Definition 4.10 (Validity, **BDL**)**.** *For all $A \in \mathcal{L}_D$,*

$$\vDash_{\mathbf{BDL}} A \text{ iff for all } \mathbf{BDL} \text{ models } \mathcal{M}\text{: } \mathcal{M} \vDash A.$$

Definition 4.11 (Logical Consequence, **BDL**)**.** *For all $\Phi \subseteq \mathcal{L}_D$ and $A \in \mathcal{L}_D$,*

$$\Phi \vDash_{\mathbf{BDL}} A \text{ iff for all } \mathbf{BDL} \text{ models } \mathcal{M}, \text{ if } \mathcal{M} \vDash \Phi, \text{ then } \mathcal{M} \vDash A.$$

This finally gives us a semantics for full **BDL**.

We now conclude this section with a few remarks on (DDS). Although our general truthmaker framework *can* give us a semantics for full **BDL**, we are not convinced that a conflict tolerant logic *should* contain (DDS) (at least not in its general form). Goble argues for (DDS) by referring to examples of intuitively valid inferences that are warranted if a logic contains (DDS). However, the following three simple observations might make one skeptical about (DDS)'s initial plausibility. The first two arguments merely point out oddities that are independent of our semantics. The third argument, however, shows that (DDS) is incompatible with a property we might want the set Ok of admissible states to satisfy. From a logical point of view, none of the following points is entirely defeating, so we leave it to the reader to decide whether (DDS)'s initial plausibility outweighs the drawbacks we are about to point out.

First, note that the "standard" argument for (DDS) doesn't work anymore in a setting that allows for normative conflicts. The standard argument for (DDS) in an conflict *intolerant* setting goes like this: suppose that (i) $O\neg A$ and that (ii) $O(A \vee B)$. Since an intolerant setting rules out normative conflicts, we cannot have OA, i.e. (iii) $\neg OA$, and hence the only reason reason for $O(A \vee B)$ to be true has to be that (iv) OB is true. Now in the conflict *tolerant* setting, we cannot any longer make the step from (i) to (iii), which blocks the (logical) standard argument in the favor of (DDS).

The second point is due to Frederik van de Putte, and it consists in showing that, in very specific cases, (DDS) seems to generate "new" conflicts: suppose that we have $O(A \vee (B \wedge \neg B))$ and $O\neg A$. Applying (DDS) gives us $O(B \wedge \neg B)$, which is equivalent to $OB \wedge O\neg B$. As was to be expected, we have a normative conflict with respect to B. However $O(B \wedge \neg B)$ also implies $O(B \vee \neg B)$, which in turn gives us $O\neg(B \wedge \neg B)$.[12] If we use (DDS) once more with this and the initial assumption $O(A \vee (B \wedge \neg B))$, we can conclude that OA. Since we also had $O\neg A$, we now ended up with another (a new one?) conflict with respect to A.

The third argument makes use of our semantics. As we have seen in one of the examples at the end of section 3, the Ok set is not closed under subsets. So an Ok set of a **BDL** model does not have to satisfy the following closure property:

Definition 4.12 (Closure under subsets)**.** *A set of states* Σ *is* closed under subsets *iff for all* $\sigma \in \Sigma$*, we also have* $\sigma' \in \Sigma$*, whenever* $\sigma' \subseteq \sigma$*.*

That Ok sets do not have to satisfy closure under subset means that a state can be admissible without all of its parts being admissible. But this seems odd: how can a complex state be admissible and still contain a part that is not admissible? So it seems only natural to further restrict Ok such that situations like these are excluded, i.e. to require Ok sets to be closed under subsets. As natural as this might look at first sight, it has disastrous consequences if combined with (DDS): as soon as we require Ok sets to be closed under subsets, we end up with an extension of **BDL** that deontically explodes in situations of normative conflicts. To see this note that closing Ok under subsets results in the validity of

[12] Due to (RBE) and the fact that analytic equivalence satisfies De Morgan's laws.

(Add) $OA \to O(A \vee B)$.[13]

It is well-known that (Add) doesn't (always) do well in the presence of (DDS) and normative conflicts.[14] The following argument is well-known, and it is as simple as it is effective: suppose that there is a normative conflict, i.e. OA and $O\neg A$. It's obvious that OA, (Add) and (MP) result in $O(A \vee B)$, which together with $O\neg A$ and (DDS) results in OB, for arbitrary B. Hence, the logic deontically explodes. And now the dilemma is apparent: we cannot have (DDS) and make Ok sets closed under subsets. Again, this argument does not conclusively show which way one should go. However, where Goble provides reasons for (DDS) in terms of intuitively valid natural language examples, we give strong semantic reasons in favor of closing Ok under subsets. So if we had to pick an option at this stage of the inquiry, we'd rather base our decision on reasons of the latter sort.

5 Main Results: Soundness & Completeness of BDL$^-$ and BDL

In this section, we set out to prove the soundness and completeness of our semantics with respect to **BDL**$^-$. The soundness and completeness of the semantics for **BDL** then follows as a corollary from the soundness and completeness result and Theorem 4.8.

We start with soundness and observe the following lemma:

Lemma 5.1. *The following are equivalent for all* **BDL**$^-$ *models* $\mathcal{M} = (\omega, Ok)$*:*

1. $[\![A \wedge B]\!]^- \cap Ok = \emptyset$
2. $[\![A]\!]^- \cap Ok = \emptyset$ *and* $[\![B]\!]^- \cap Ok = \emptyset$

Proof. 1. $\Rightarrow$ 2. Remember that

$$[\![A \wedge B]\!]^- = conv(\underbrace{[A \wedge B]^-}_{=[A]^- \cup [B]^- \cup [A \vee B]^-}).$$

Since $[\![A]\!]^- = conv([A]^-)$ and $[\![B]\!]^- = conv([B]^-)$, we get that $[\![A]\!]^-, [\![B]\!]^- \subseteq [\![A \wedge B]\!]^-$. So, if $[\![A \wedge B]\!]^- \cap Ok = \emptyset$, then also both $[\![A]\!]^- \cap Ok = \emptyset$ and $[\![B]\!]^- \cap Ok = \emptyset$.

2. $\Rightarrow$ 1. To prove this direction, we need to use both conditions for **BDL**$^-$ models, reverse convexity and reverse closure. Suppose that $[\![A]\!]^- \cap Ok = \emptyset$ and $[\![B]\!]^- \cap Ok = \emptyset$. And suppose further that $\sigma \in [\![A \wedge B]\!]^-$. Since

$$[\![A \wedge B]\!]^- = [A]^- \cup [B]^- \cup [A \vee B]^- \cup \{\sigma : \exists \tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-, \tau \subseteq \sigma \subseteq \pi\},$$

we can distinguish four cases: (i) $\sigma \in [\![A]\!]^-$, (ii) $\sigma \in [\![B]\!]^-$, (iii) $\sigma \in [\![A \vee B]\!]^-$, and (iv) $\sigma \in \{\sigma : \exists \tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-, \tau \subseteq \sigma \subseteq \pi\}$. Cases (i) and (ii) are directly excluded by our assumption that $[\![A]\!]^- \cap Ok = \emptyset$ and $[\![B]\!]^- \cap Ok = \emptyset$. For case (iii), assume that $\sigma \in [\![A \vee B]\!]^-$, i.e. there are τ and π such that $\sigma = \tau \cup \pi$ with $\tau \in [\![A]\!]^-$ and $\pi \in [\![B]\!]^-$. If there are such τ and π, we know that $\tau, \pi \notin Ok$

[13] See also the first example at the end of section 3.

[14] Goble also makes this observation in [11].

by assumption, so we can conclude that $\tau \cup \pi = \sigma \notin Ok$ by the reverse closure property of Ok. Finally, for $\sigma \in \{\sigma : \exists \tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-, \tau \subseteq \sigma \subseteq \pi\}$, note that we've already seen in cases (i–iii) that all $\tau, \pi \in [A]^- \cup [B]^- \cup [A \vee B]^-$ are not in Ok. So $\sigma \notin Ok$ by the reverse convexity property of Ok. □

In combination with our previous observations, this lemma gives us the soundness of our semantics for $\mathbf{BDL}^-$:

Theorem 5.2 (Soundness for $\mathbf{BDL}^-$)**.** *For all* $\Gamma \subseteq \mathcal{L}_D$ *and* $A \in \mathcal{L}_D$*,*

$$\Gamma \vdash_{\mathbf{BDL}^-} A \Rightarrow \Gamma \vDash_{\mathbf{BDL}^-} A.$$

Proof. By Lemma 3.15, all substitution instances of classical tautologies are true in all $\mathbf{BDL}^-$ models. By Lemma 5.1, (M) and (AGG) are true in all $\mathbf{BDL}^-$ models. (MP) preserves truth in a $\mathbf{BDL}^-$ models by a standard argument. And by Theorem 3.7, (RBE) preserves truth in $\mathbf{BDL}^-$ models. □

Next we turn to completeness. For this, we need a few auxiliary concepts. Remember that a *literal* is either an atom $p \in \mathcal{A}$ or the negation $\neg p$ of an atom $p \in \mathcal{A}$. In the following, we'll use λ (possibly indexed) as a meta-variable for literals.

Remember further that a formula is in *conjunctive normal form* iff it is a conjunction of disjunction of literals:

Definition 5.3. *A formula* $A \in \mathcal{L}$ *is in* conjunctive normal form *(CNF) iff*

$$A = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})$$

for literals λ_i^j *and* f *a function that gives us the length of the* i*-th conjunct.*

In standard propositional logic, we can show that every formula is equivalent to a formula in CNF. This is the CNF Theorem. We have the following truthmaker version of this result, which essentially says that every formula is *analytically* equivalent to a formula in CNF:

Theorem 5.4 (Analytic CNF Theorem)**.** *For every formula* $A \in \mathcal{L}$*, there is a formula* B *in conjunctive normal form, such that both*

1. $[\![A]\!]^+ = [\![B]\!]^+$ *and*
2. $[\![A]\!]^- = [\![B]\!]^-$.

Proof. By induction on the complexity of A, as in the standard proof of the CNF Theorem. Note that all the transformations in the standard argument are also analytic equivalences. Note in particular that we have $\vdash_{AC} A \vee (B \wedge C) \Leftrightarrow_A (A \vee B) \wedge (A \vee C)$, which essentially relies on the fact that we're using replete contents! By Theorem 3.7, we get the desired result. □

Literals correspond in a natural way to the "building blocks" of our states, i.e. pairs of the form (p, x), where $p \in \mathcal{A}$ and $x \in \{0, 1\}$:

Definition 5.5 (Trace states)**.** *Let* λ *be a literal. Then we define:*

$$tr(\lambda) := \begin{cases} (p, 1) & \text{if } \lambda = p \\ (p, 0) & \text{if } \lambda = \neg p \end{cases}.$$

Now, to the completeness argument. The argument proceeds in a more or less standard fashion by means of a canonical model construction via maximally consistent sets.

Definition 5.6. *A set of formulas Φ is consistent iff there is no A such that $\Phi \vdash A$ and $\Phi \vdash \neg A$.*

Definition 5.7. *A set of formulas Φ is maximally consistent (with respect to* $\mathbf{BDL}^-$*) iff both of the following hold:*

1. *Φ is consistent*
2. *for all consistent sets $\Phi' \supseteq \Phi$, $\Phi = \Phi'$.*

By a standard argument, we can show that every consistent set of formulas can be extended to a maximally consistent set:

Theorem 5.8. *For any consistent set of formulas Φ there is a maximally consistent set of formulas Φ^* sucht that $\Phi \subseteq \Phi^*$.*

Note that maximally consistent set have the following canonical properties:

Lemma 5.9. *If Φ is maximally consistent, then for all A,*

1. *either $A \in \Phi$ or $\neg A \in \Phi$ and never both*
2. $A \in \Phi \Leftrightarrow \Phi \vdash A$
3. $A \notin \Phi \Leftrightarrow \Phi \vdash \neg A$
4. $\Phi \vdash A \Leftrightarrow A \in \Phi$

A maximally consistent set determines a canonical model as follows:

Definition 5.10 (Canonical Model)**.** *Let Φ be a maximally consistent set of formulas. We define the canonical* $\mathbf{BDL}^-$ *model $\mathcal{M}_\Phi = (\omega_\Phi, Ok_\Phi)$ for Φ as follows:*

(i) $\omega_\Phi(p) = \begin{cases} 1 & \text{if } p \in \Phi \\ 0 & \text{if } p \notin \Phi \end{cases}$

(ii) $Ok_\Phi = \{\overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}} : \Phi \nvdash O(\lambda_1 \vee \ldots \vee \lambda_n)\}$

For the remainder of the proof it's useful to note that any state σ can be written as $\overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}}$ for suitable literals $\lambda_1, \ldots, \lambda_n$. In the following, we make free use of this observation without much further notice.

We now check that $\mathcal{M}_\Phi$ is indeed a $\mathbf{BDL}^-$ model:

Lemma 5.11. *For Φ a maximally consistent set of formulas and $\mathcal{M}_\Phi = (\omega_\Phi, Ok_\Phi)$ as defined in Definition 5.10:*

1. *ω_Φ is a valuation,*
2. *Ok_Φ is reverse convex,*
3. *Ok_Φ is reverse closed.*

Proof.

1. Since for all p, either $p \in \Phi$ or $p \notin \Phi$ and never both, we know that ω_Φ is a well-defined function from $\mathcal{A}$ to $\{0,1\}$.

2. To see that Ok_Φ is reverse convex, suppose that $\tau, \pi \notin Ok_\Phi$ and $\tau \subseteq \sigma \subseteq \pi$. This means that:

   - $\tau = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}}$
   - $\sigma = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n), \ldots, tr(\lambda_{n+m})\}}$
   - $\pi = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n), \ldots, tr(\lambda_{n+m}), \ldots, tr(\lambda_{n+m+k})\}}$

   with

   - $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$
   - $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$

   To see this note that $\overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}} \notin \{\overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}} : \Phi \nvdash O(\lambda_1 \vee \ldots \vee \lambda_n)\}$ just in case it's not the case that $\Phi \nvdash O(\lambda_1 \vee \ldots \vee \lambda_n)$, meaning precisely $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$.

   We want to show that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m})$. Since $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$ and $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$ we can infer via (AGG) that $\Phi \vdash O((\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda_1 \vee \ldots \vee \lambda_{n+m+k}))$. It is somewhat tedious but possible to show in *AC* that $(\lambda_1 \vee \ldots \vee \lambda_{n+m}) \wedge (\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$ is analytically equivalent to $(\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda_1 \vee \ldots \vee \lambda_{n+m+k})$. By a single application of (KBE) (see §2) we get $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_{n+m})$, meaning $\sigma \notin Ok_\Phi$ as desired.

3. To show that Ok_Φ is reverse closed, suppose that $\sigma, \tau \notin Ok_\Phi$, meaning:

   - $\sigma = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n)\}}$
   - $\tau = \overline{\{tr(\lambda'_1), \ldots, tr(\lambda'_m)\}}$

   with

   - $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$
   - $\Phi \vdash O(\lambda'_1 \vee \ldots \vee \lambda'_m)$.

   We want to show that that $\sigma \cup \tau \notin Ok_\Phi$, meaning that

   $$\sigma \cup \tau = \overline{\{tr(\lambda_1), \ldots, tr(\lambda_n), tr(\lambda'_1), \ldots, tr(\lambda'_m)\}}$$

   is such that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n \vee \lambda'_1 \vee \ldots \vee \lambda'_m)$. Since we know that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n)$ and $\Phi \vdash O(\lambda'_1 \vee \ldots \vee \lambda'_m)$, we can conclude via (AGG) that $\Phi \vdash O((\lambda_1 \vee \ldots \vee \lambda_n) \wedge (\lambda'_1 \vee \ldots \vee \lambda'_m))$. Since we know that $\vdash O(A \wedge B) \rightarrow O(A \vee B)$, we can infer that $\Phi \vdash O(\lambda_1 \vee \ldots \vee \lambda_n \vee \lambda'_1 \vee \ldots \vee \lambda'_m)$, as desired.

□

Note that Φ being maximally consistent doesn't play a role for the fact that Ok_Φ is reverse convex and reverse closed. It does play a role, however, in showing that $\mathcal{M}_\Sigma$ has the canonical model property:

Lemma 5.12 (Truth lemma)**.** *If Φ is maximally consistent, then*

$$\mathcal{M}_\Phi \vDash A \Leftrightarrow A \in \Phi.$$

Proof. By induction on complexity of the members of A:

1. If $p \in \Phi$, then $\omega_\Phi(p) = 1$ by definition, and so $\mathcal{M}_\Phi \vDash p$ as desired. If $p \notin \Phi$, then $\omega_\phi(p) = 0$ and so $\mathcal{M}_\Phi \nvDash p$ as desired.

2. If $\neg A \in \Phi$, then $A \notin \Phi$, since Φ is consistent. Hence $\mathcal{M}_\Phi \nvDash A$ by the induction hypothesis and so $\mathcal{M}_\Phi \vDash \neg A$. If $\neg A \notin \Phi$, then $A \in \Phi$, since Φ is maximally consistent. Hence $\mathcal{M}_\Phi \vDash A$ and so $\mathcal{M}_\Phi \nvDash \neg A$ as desired.

3. If $A \wedge B \in \Phi$, then, since $A \wedge B \vdash A$ and $A \wedge B \vdash B$ and Φ is maximally consistent, we get that $A, B \in \Phi$. So $\mathcal{M}_\Phi \vDash A$ and $\mathcal{M}_\Phi \vDash B$ by the induction hypothesis and so $\mathcal{M}_\Phi \vDash A \wedge B$ as desired. If $A \wedge B \notin \Phi$, then $\neg(A \wedge B) \in \Phi$ since Φ is maximally consistent. Next we show that $\neg A \in \Phi$ or $\neg B \in \Phi$. For if both $\neg A \notin \Phi$ and $\neg B \notin \Phi$, then, since Φ is maximally consistent, we'd have $A, B \in \Phi$. But then we'd get $\Phi \vdash A \wedge B$, and, since $\neg(A \wedge B) \in \Phi$, $\Phi \vdash \neg(A \wedge B)$, making Φ inconsistent. So $\neg A \in \Phi$ or $\neg B \in \Phi$. So either $A \notin \Phi$ of $B \notin \Phi$. So either $\mathcal{M}_\Phi \nvDash A$ or $\mathcal{M}_\Phi \nvDash B$. Hence $\mathcal{M}_\Phi \nvDash A \wedge B$ as desired.

4. The cases for $A \vee B \in \Phi$ and $A \vee B \notin \Phi$ are analogous to the previous case.

5. The main cases are (a) $OA \in \Phi$ and (b) $OA \notin \Phi$.

   (a) Suppose that $OA \in \Phi$. Then, of course, $\Phi \vdash OA$. Now by the analytic CNF Theorem 5.4, there is a B in CNF, such that $[\![A]\!]^+ = [\![B]\!]^+$ and $[\![A]\!]^- = [\![B]\!]^-$. By Theorem 3.7, we get $\vdash_{AC} A \Leftrightarrow_A B$ and so by (RBE) $\Phi \vdash OB$. Let

   $$B = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}),$$

   where f is a function that gives us the length of the i-th conjunct of B. By (M), we get

   $$\Phi \vdash O(\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)})$$

   for all $1 \leq i \leq n$. By definition of Ok_Φ this means that

   $$\overline{\{tr(\lambda_i^1), \ldots, tr(\lambda_i^{f(i)})\}} \notin Ok_\Phi.$$

   It is easily checked that $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- = \{\overline{\{tr(\lambda_i^1), \ldots, tr(\lambda_i^{f(i)})\}}\}$ and so $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- \cap Ok_\Phi = \emptyset$ for all $1 \leq i \leq n$. By repeated application of Lemma 5.1, we get

   $$[\![(\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})]\!]^- \cap Ok_\Phi = \emptyset,$$

   which just means that $\mathcal{M}_\Phi \vDash O((\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}))$ as desired.

(b) Suppose $OA \notin \Phi$. Then, since Φ is maximally consistent, $\Phi \nvdash OA$. Again by the analytic CNF Theorem 5.4 and Theorem 3.7, such that $\vdash_{AC} A \Leftrightarrow_A B$. We can conclude that also $\Phi \nvdash OB$. For if $\Phi \vdash OB$, then by (RBE) we'd get $\Phi \vdash OA$ and so $OA \in \Phi$, since Φ is maximally consistent. Contradiction. Hence $\Phi \nvdash OB$. Now let again

$$B = (\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}),$$

where f is again a function that gives us the length of the i-th conjunct of B. Next we conclude that $\Phi \nvdash O(\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)})$, for some $1 \leq i \leq n$. For if $\Phi \vdash O(\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)})$ for all $1 \leq i \leq n$, we'd get $\Phi \vdash OB$ by (AGG). Since $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- = \{\{tr(\lambda_i^1), \ldots, tr(\lambda_i^{f(i)})\}\}$ and $Ok_\Phi = \{\{tr(\lambda_1), \ldots, tr(\lambda_n)\} : \Phi \nvdash O(\lambda_1 \vee \ldots \vee \lambda_n)\}$, we can conclude that $[\![\lambda_i^1 \vee \ldots \vee \lambda_i^{f(i)}]\!]^- \cap Ok_\Phi \neq \emptyset$. By Lemma 5.1, we can conclude that

$$[\![(\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)})]\!]^- \cap Ok_\Phi \neq \emptyset,$$

which just means that $\mathcal{M}_\Phi \nvDash O((\lambda_1^1 \vee \ldots \vee \lambda_1^{f(1)}) \wedge \ldots \wedge (\lambda_k^1 \vee \ldots \vee \lambda_n^{f(n)}))$ as desired.

□

The rest of the proof is routine:

Lemma 5.13. *If $\Phi \nvdash A$, then $\Phi \cup \{\neg A\}$ is consistent.*

Theorem 5.14 (Completeness for $\mathbf{BDL}^-$)**.** *For all $\Phi \subseteq \mathcal{L}_D$ and $A \in \mathcal{L}_D$,*

$$\Phi \vDash_{\mathbf{BDL}^-} A \Rightarrow \Phi \vdash_{\mathbf{BDL}^-} A.$$

Proof. We prove the contrapositive. So suppose that $\Phi \nvdash A$. Then $\Phi \cup \{\neg A\}$ is consistent by Lemma 5.13. Extend $\Phi \cup \{\neg A\}$ to a maximally consistent set $(\Phi \cup \{\neg A\})^* \supseteq \Phi \cup \{\neg A\}$ using Theorem 5.8. Consider $\mathcal{M}_{(\Phi \cup \{\neg A\})^*}$. By Lemma 5.12, we get $\mathcal{M}_{(\Phi \cup \{\neg A\})^*} \vDash \Phi \cup \{\neg A\}$. So, in particular, $\mathcal{M}_{(\Phi \cup \{\neg A\})^*} \vDash \Phi$ and $\mathcal{M}_{(\Phi \cup \{\neg A\})^*} \nvDash A$, giving us that $\Phi \nvDash A$.

□

This concludes our proof of the soundness and completeness of $\mathbf{BDL}^-$.

As advertised, the soundness and completeness of **BDL** follows as a corollary from this result:

Corollary 5.15 (Soundness and Completeness of **BDL**)**.** *For all $\Phi \subseteq \mathcal{L}_D$ and $A \in \mathcal{L}_D$,*

$$\Phi \vDash_{\mathbf{BDL}} A \Leftrightarrow \Phi \vdash_{\mathbf{BDL}} A.$$

Proof. We have $\Phi \vdash_{\mathbf{BDL}} A$ iff $\Phi \cup \{O(A \vee B) \wedge O\neg A \rightarrow OB : A, B \in \mathcal{L}\} \vdash_{\mathbf{BDL}^-} A$. By the soundness and completeness for $\mathbf{BDL}^-$, we have that $\Phi \cup \{O(A \vee B) \wedge O\neg A \rightarrow OB : A, B \in \mathcal{L}\} \vdash_{\mathbf{BDL}^-} A$ iff $\Phi \cup \{O(A \vee B) \wedge O\neg A \rightarrow OB : A, B \in \mathcal{L}\} \vDash_{\mathbf{BDL}^-} A$. By Theorem 4.8, we know that $\Phi \cup \{O(A \vee B) \wedge O\neg A \rightarrow OB : A, B \in \mathcal{L}\} \vDash_{\mathbf{BDL}^-} A$ iff $\Phi \vDash_{\mathbf{BDL}} A$

□

6 Conclusion and Outlook

The purpose of Goble's systems **BDL** and **BDL**$^-$ was to serve as a logic for a notion of obligation that allows for normative conflicts. So in Goble's paper, the main purpose of these systems was not to (excplicitly, at least) also provide a logic other deontic notions (like permissions, prohibitions). If, however, **BDL** and **BDL**$^-$ are good logics for dealing with normative conflicts, these systems might just as well serve as a base for full-blown deontic logics. And it's actually not that difficult to also interpret other deontic notions in this framework. For example, as we have already hinted at in the introduction, the set of admissible states also allows for natural truth-conditions for permissions. For instance, one could take a **BDL**$^-$ model $\mathcal{M} = (\omega, Ok)$ and add the following truth-condition for permissions:[15]

Definition 6.1 (Truth, Permissions)**.**

(vi) $\mathcal{M} \vDash PA$ *iff* $[\![A]\!]^+ \subseteq Ok$

This truth-condition results in a hyperintensional logic of permission. If the Ok set is further closed under union, it validates (full) free choice permission (FCP), i.e. $P(A \vee B) \leftrightarrow PA \wedge PB$. But due to its hyperintensional character, it avoids (at least most, if not all) the problematic consequences any intensional logic of FCP is accompanied by. In addition to that, it yields interesting interaction principles between obligations and permissions. For example, it's easily checked that the semantics results in $\vDash OA \rightarrow \neg P \neg A$, but not the other way round, i.e. $\nvDash \neg P \neg A \rightarrow OA$. This is all we wanted to say at this point, and we leave the study of these full-blown deontic logics for another day.

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