

Yet Another Puzzle of Ground*



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Abstract

We show that any predicational theory of partial ground that extends a standard theory of syntax and that proves some commonly accepted principles for partial ground is inconsistent. We suggest a way to obtain a consistent predicational theory of ground.

Keywords: *grounding, standard theory of ground*

Fine defines ground as “the relation of one truth holding in virtue of others” [7, p. 1].¹ Given this definition, it is natural to think that we should formulate axiomatic first-order theories of ground, which formalize ground by means of a relational *ground predicate* of true sentences. Call such theories *predicational theories of ground*. Predicational theories of ground contrast with *operational theories of ground*, which formalize ground by means of a sentential *ground operator* [3, p. 253–54, 5, p. 46–47]. So far, most theories of ground in the literature are operational theories of ground. But there are at least three theoretical reasons for developing predicational theories of ground:

1. *Quantification*: Predicational theories of ground have greater expressive strength than operational theories of ground. In particular, using a ground predicate, we can formalize ground-theoretic principles involving quantification over truths in a natural way. Take, for example, the intuitively plausible claim that every truth is either fundamental or grounded in some other truths. We can straightforwardly formalize this claim using a ground predicate and first-order quantification over truths, but using a ground operator this is impossible. Without the use of non-classical devices, such

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as propositional quantification, it is impossible to formalize the nested universal and existential quantification over truths in the principle. Using a ground predicate, in contrast, we can formalize the principle comfortably in the purview of classical first-order logic.

2. *Truth and Modality*: Predicational theories of ground allow us to study ground in the same context as truth and modality. It is generally accepted that truth should be treated as a predicate of sentences, and it has recently been suggested to extend this approach to modality as well [10, 13, 8]. There is an obvious connection between ground and truth, since ground is a relation among truths. But there is also a close connection between ground and modality, since ground is usually assumed to imply necessary consequence: if a truth holds in virtue of some other truths, then the former truth should be a necessary consequence of the latter truths [5, p. 38–39]. Both of these connections are most naturally studied using predicational theories of ground: by combining predicational theories of ground with predicational theories of truth and modality.
3. *Models*: Predicational theories of ground allow us to discover and to study models of ground using classic model-theoretic methods. It is currently an open problem to provide a semantics for the *impure logic of ground* developed by Fine [5, p. 58–71]. This logic is formulated using a ground operator, but once we translate it into a predicational theory of ground and show its consistency, we can rely on model-theoretic theorems to establish the existence of first-order models. Once we know that such models exist, we can study them using methods of model theory. This should provide us with new insights into the semantics of the impure logic of ground.

But predicational theories of ground face a paradox of self-reference, similar to the well-known paradoxes of self-reference that arise in predicational theories of truth and modality. In this paper, I shall prove this point for predicational theories of *partial ground* in particular. This is the relation of one truth holding partially in virtue of another truth—the relation of one truth “helping” to ground another truth [5, p. 50]. I show that any predicational theory of partial ground that extends a standard theory of syntax and that proves some commonly accepted principles for partial ground is inconsistent. Fine [6] and Krämer [11] present puzzles about the irreflexivity of partial ground: the principle that no truth partly grounds itself. They show that certain intuitively

plausible principles of logic and metaphysics lead to counterexamples to the irreflexivity of ground. I add yet another puzzle of ground to the mix. The new puzzle does not mention the irreflexivity of ground or metaphysical principles unrelated to ground, thus it is genuinely different from the previously known paradoxes.

To formulate a predicational theory of partial ground, we first need a theory of syntax that allows us to talk about sentences.² It is well-known that we can develop such a theory in any sufficiently strong background theory, like Robinson arithmetic for example. For the present purpose, however, our concrete choice of background theory does not matter. All that matters is that our background theory Θ satisfies the following three minimal syntax conditions:³ The first condition is that Θ proves that we have a unique name $\ulcorner \varphi \urcorner$ for every sentence φ in the sense that for all sentences φ and ψ , $\Theta \vdash \ulcorner \varphi \urcorner = \ulcorner \psi \urcorner$ only if $\varphi = \psi$. The second condition is that Θ proves that we have a function symbol $\underline{\vee}$ that represents the syntactic operation \vee of disjunction in the sense that for all sentences φ and ψ , $\Theta \vdash \ulcorner \varphi \urcorner \underline{\vee} \ulcorner \psi \urcorner = \ulcorner \varphi \vee \psi \urcorner$. And the third condition is that Θ proves the diagonal lemma. Informally, this lemma states that for every condition on sentences there is a sentence that is provably equivalent to the condition holding of itself. More precisely, if $\varphi(x)$ is a formula with exactly one free variable, then there exists a sentence δ such that $\Theta \vdash \delta \leftrightarrow \varphi(\ulcorner \delta \urcorner)$. Note that any standard background theory of syntax, such as Robinson arithmetic, satisfies all three of our minimal syntax conditions.

Next, we need a way of representing partial ground. For this purpose, we use the relational predicate $x \triangleleft y$. For sentences φ and ψ , we informally read the atomic formula $\ulcorner \varphi \urcorner \triangleleft \ulcorner \psi \urcorner$ as saying that the truth of φ partially grounds the truth of ψ . For a negated atomic formula of the form $\neg(\ulcorner \varphi \urcorner \triangleleft \ulcorner \psi \urcorner)$ we also write $\ulcorner \varphi \urcorner \not\triangleleft \ulcorner \psi \urcorner$, which we correspondingly read as saying that the truth of φ does not even partially ground the truth of ψ .

Philosophers have laid down various principles for partial ground [cf. 17, 5, 7], but it is already sufficient for a predicational theory of partial ground to be inconsistent that it proves two widely accepted principles. Let Θ now be a predicational theory of partial ground that satisfies the minimal syntax conditions. The first of our two principles follows directly from partial ground being a relation of true sentences: If the truth of one sentence partially grounds the truth of another, then both sentences should be true. This principle is known as the “factivity of ground” and is generally accepted in the literature on ground [6, p. 100, 1, § 3].⁴ We get the condition on Θ that for all sentences φ and ψ :

$$\begin{aligned} (\text{Fact}_L): \quad & \Theta \vdash \ulcorner \varphi \urcorner \triangleleft \ulcorner \psi \urcorner \rightarrow \varphi \\ (\text{Fact}_R): \quad & \Theta \vdash \ulcorner \varphi \urcorner \triangleleft \ulcorner \psi \urcorner \rightarrow \psi \end{aligned}$$

The second principle concerns the interaction of partial ground and disjunction: Given that partial ground is the relation of one truth holding partially in virtue of another, if a disjunction is true, then its truth should be partially grounded in each of its true disjuncts. Also this principle is generally accepted in the literature on ground [6, p. 101, 17, p. 117]. From this, we get the condition on Θ that for all sentences φ and ψ :

$$\begin{aligned} (\vee_1): \quad & \Theta \vdash \varphi \rightarrow \ulcorner \varphi \urcorner \triangleleft \ulcorner \varphi \vee \psi \urcorner \\ (\vee_2): \quad & \Theta \vdash \psi \rightarrow \ulcorner \psi \urcorner \triangleleft \ulcorner \varphi \vee \psi \urcorner \end{aligned}$$

The minimal syntax conditions and the conditions concerning partial ground all may seem fairly uncontroversial when viewed individually. So it may be somewhat surprising to learn that there can be no consistent predicational theory of partial ground that satisfies all of them:

Theorem (Inconsistency Theorem). *Any theory Θ that satisfies the minimal syntax conditions, $(\text{Fact}_{L/R})$, and $(\vee_{1/2})$ is inconsistent.*

Proof. Let $\varphi(x)$ be the formula $x \not\triangleleft x \vee x$. By the diagonal lemma, there is a sentence δ such that $\Theta \vdash \delta \leftrightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \urcorner \vee \ulcorner \delta \urcorner$. Intuitively, this is a sentence which “says of itself” that it does not partially ground its own disjunction. By the second minimality condition, we have that $\Theta \vdash \ulcorner \delta \urcorner \vee \ulcorner \delta \urcorner = \ulcorner \delta \vee \delta \urcorner$. From this and $\Theta \vdash \delta \leftrightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \urcorner \vee \ulcorner \delta \urcorner$, we get that $\Theta \vdash \delta \leftrightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner$ by the substitutivity of identicals. This splits up into the following two conditions:

- (a) $\Theta \vdash \delta \rightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner$
- (b) $\Theta \vdash \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner \rightarrow \delta$

We get finally the following argument:

1. $\Theta \vdash (\ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner \rightarrow \delta) \rightarrow ((\delta \rightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner) \rightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner)$
(Tautology[†])
2. $\Theta \vdash \ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner \rightarrow \delta$ (Fact_L)
3. $\Theta \vdash (\delta \rightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner) \rightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner$ (1, 2: MP)
4. $\Theta \vdash \delta \rightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner$ (a)
5. $\Theta \vdash \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner$ (3, 4: MP)
6. $\Theta \vdash \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner \rightarrow \delta$ (b)
7. $\Theta \vdash \delta$ (5, 6: MP)

$$8. \Theta \vdash \delta \rightarrow \ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner \quad (\vee_1)$$

$$9. \Theta \vdash \ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner \quad (7,8: \text{MP})$$

$$10. \Theta \vdash \perp \quad (5,9: \perp)$$

(†) : Note that every sentence of the form $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \neg\varphi) \rightarrow \neg\varphi)$ is a classical tautology and that theories prove all classical tautologies. \square

The inconsistency theorem is very similar to Tarski's theorem about predicational theories of truth [18] and Montague's theorem about predicational theories of modality [15] in that it is, essentially, a paradox of self-reference. From a technical perspective, it should in fact not be surprising that we get such a theorem after all: Combining self-reference via the diagonal lemma with principles like (Fact_{L/R}) that allow us to push a sentence outside the scope of a predicate and principles like ($\vee_{1/2}$) that allow us to push a sentence into the scope of a predicate is recipe for disaster.⁵ But from a philosophical perspective, there is a lesson to be learned: We already know that we cannot understand ground simply in terms of truth and modality [5], but ground behaves syntactically too much like a combination of truth and modality to escape inconsistency when paired with self-reference.

Three natural ways in which we could try to block the inconsistency theorem suggest themselves: First, we could try to rule out self-referential sentences of ground like the one used in the proof of the inconsistency theorem. Second, we could try to restrict the principles of partial ground used in the proof of the inconsistency theorem. And third, we could try to formulate a non-standard logic of ground that does not sanction the logical principles used in the proof of the inconsistency theorem. The analogy between the inconsistency theorem and the theorems of Tarski and Montague suggests a terminology for these approaches. Analogously to predicational theories of truth [9] and predicational theories of modality [8], we get: *typed theories of partial ground*, which avoid paradox by putting type-restrictions on the relation of partial ground, effectively ruling out self-referential sentences like the one in the proof; *untyped theories of partial ground*, which avoid paradox by restricting the principles of partial ground; and finally *non-classical theories of partial ground*, which avoid paradox (or: triviality) by abandoning classical logic in favor of alternative logics.

Untyped theories of partial ground are particularly appealing, because considerations of ground are already part of intuitively appealing approach to untyped theories of truth. On an influential view about

predicational theories of truth, self-referential sentences are *ungrounded* and this is the reason some self-referential sentences lead to inconsistency [12, 14]. This leads to the idea that we should restrict the principles of truth to their grounded instances.⁶ Carrying this idea from theories of truth over to predicational theories of ground, we arrive at the condition that the principles of partial ground apply if and only if the truths involved are themselves grounded. There is a straightforward way of formulating the desired restriction on the principles of partial ground already in the language of partial ground. We can express that a sentence φ is grounded by the formula $\exists x(x \triangleleft \ulcorner \varphi \urcorner)$ and we can express that a sentence φ is ungrounded by the formula $\neg \exists x(x \triangleleft \ulcorner \varphi \urcorner)$. The desired restriction on $(\vee_{1/2})$ then amounts to saying that for all predicational theories of ground Θ and for all sentences φ and ψ :

$$\begin{aligned} (\vee_1^*): \quad \Theta \vdash \exists x(x \triangleleft \ulcorner \varphi \urcorner) &\leftrightarrow \ulcorner \varphi \urcorner \triangleleft \ulcorner \varphi \vee \psi \urcorner \\ (\vee_2^*): \quad \Theta \vdash \exists x(x \triangleleft \ulcorner \psi \urcorner) &\leftrightarrow \ulcorner \psi \urcorner \triangleleft \ulcorner \varphi \vee \psi \urcorner \end{aligned}$$

Every predicational theory of partial ground that satisfies the minimal syntax conditions, $(\text{Fact}_{L/R})$, and the new conditions $(\vee_{1/2}^*)$, proves that the paradoxical sentence in the proof of the theorem is ungrounded:

Observation. *Let Θ be a predicational theory of ground that satisfies the minimal syntax conditions, $(\text{Fact}_{L/R})$, and $(\vee_{1/2}^*)$. By the diagonal lemma and the same reasoning as in the proof of the theorem, we get a sentence δ such that:*

$$\Theta \vdash \delta \leftrightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner.$$

But we can show that:

$$\Theta \vdash \neg \exists x(x \triangleleft \ulcorner \delta \urcorner).$$

Proof. By applying (\vee_1^*) to δ , we get that:

$$\Theta \vdash \exists x(x \triangleleft \ulcorner \delta \urcorner) \leftrightarrow \ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner$$

We only need the “left-to-right direction” of this biconditional for our proof, which we can obtain via \leftrightarrow -Elimination:

$$\Theta \vdash \exists x(x \triangleleft \ulcorner \delta \urcorner) \rightarrow \ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner$$

Starting from there, we get the following argument:

1. $\Theta \vdash \exists x(x \triangleleft \ulcorner \delta \urcorner) \rightarrow \ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner$
2. $\Theta \vdash \ulcorner \delta \urcorner \triangleleft \ulcorner \delta \vee \delta \urcorner \rightarrow \delta$ (Fact_L)

3. $\Theta \vdash \exists x(x \triangleleft \ulcorner \delta \urcorner) \rightarrow \delta$ (1,2: MP)
4. $\Theta \vdash \delta \leftrightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner$ (Diagonal Lemma)
5. $\Theta \vdash \exists x(x \triangleleft \ulcorner \delta \urcorner) \rightarrow \ulcorner \delta \urcorner \not\triangleleft \ulcorner \delta \vee \delta \urcorner$ (3, 4: \leftrightarrow -Elim)
6. $\Theta \vdash \exists x(x \triangleleft \ulcorner \delta \urcorner) \rightarrow \perp$ (1, 5: \perp -Intro)
7. $\Theta \vdash \neg \exists x(x \triangleleft \ulcorner \delta \urcorner)$ (6: \neg -Intro)

□

This result should make us optimistic about the prospects for a untyped theory of partial ground. Moreover, we could add such a untyped theory of partial ground “on top” of untyped predicational theories of truth and modality. I conjecture that an interesting, *consistent*, untyped predicational theory of ground can be developed in this way.

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Notes

- 1 For (opinionated) introductions to ground, see [4, 5]. For an overview of the recent literature on ground, see [2, 19, 16].
- 2 A sentence is a formula without any free variables.
- 3 A theory is a set of formulas that is closed under derivability: a set of formulas Θ is a theory iff (if and only if) for all formulas φ , if $\Theta \vdash \varphi$, then $\varphi \in \Theta$.
- 4 There are notions of ground in the literature that violate the factivity of ground [cf. 5, p. 48–50]. According to such *non-factive* notions, ground is a relation on sentences regardless of their truth value. Although non-factive notions of ground make for an interesting theoretical possibility, in this paper we shall deal only with the standard *factive* notion of ground, which satisfies the factivity of ground.
- 5 It should be clear at this point that not much depends on the concrete condition $(\vee_{1/2})$ —the paradox is not a paradox of disjunction. All that matters is that our predicational theory of partial ground proves a principle to the effect that any true sentence partially grounds some other sentence. We could give the following variant of the inconsistency theorem: If Θ satisfies $(\text{Fact}_{L/R})$ and either $\Theta \vdash \varphi \rightarrow \exists x(\ulcorner \varphi \urcorner \triangleleft x)$ or $\Theta \vdash \varphi \rightarrow \exists x(x \triangleleft \ulcorner \varphi \urcorner)$, then Θ is inconsistent. I leave the details of the proof to the interested reader.
- 6 The concept of ground used in the context of theories of truth is not exactly the same as the concept of ground discussed in this paper. For example, the notion of dependence defined by Leitgeb [14] is reflexive, whereas (partial) ground is standardly taken to be irreflexive. The point here is that there is a striking

analogy between the two concepts and that ideas that work for the one may very well work for the other.

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