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# What Are Structural Properties?[†]

Johannes Korbmacher[*] and Georg Schiemer[**]

[*]Department of Philosophy and Religious Studies, Utrecht University, The Netherlands
E-mail: j.korbmacher@uu.nl
[**]Department of Philosophy, University of Vienna, Vienna, Austria
E-mail: georg.schiemer@univie.ac.

**ABSTRACT**

Informally, structural properties of mathematical objects are usually characterized in one of two ways: either as properties expressible purely in terms of the primitive relations of mathematical theories, or as the properties that hold of all *structurally similar* mathematical objects. We present two formal explications corresponding to these two informal characterizations of structural properties. Based on this, we discuss the relation between the two explications. As will be shown, the two characterizations do *not* determine the same class of mathematical properties. From this observation we draw some philosophical conclusions about the possibility of a 'correct' analysis of structural properties.

## 1. INTRODUCTION

Structural properties play a central role in the contemporary philosophy of mathematics, particularly in the debate about mathematical structuralism. This is the view that mathematics is not concerned with the 'internal nature' of its objects, but rather with how these objects 'relate to each other' [Shapiro, 1997; Resnik, 1997; Parsons, 1990]. Take the natural numbers as an example. The standard mathematical theory of the natural numbers is second-order *Peano arithmetic*. It is well-known that many different set-theoretic systems satisfy the axioms of Peano arithmetic, such as the (finite) *von Neumann* ordinals $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$ and the *Zermelo* ordinals $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$ for example. According to the structuralists, however, mathematics is not concerned with

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the concrete set-theoretic structure of these models — second-order Peano arithmetic does not describe the numbers as specific sets. Rather, arithmetic describes how the numbers add up, how they can be divided, and so on — it describes the *structure* that the set-theoretic systems satisfying the theory have in common. In other words, according to structuralism, mathematics is concerned with the *structural properties* of its objects.

Despite the importance of structural properties for mathematical structuralism, there appears to be no formal explication of the concept in the literature. Informally, structuralists usually characterize structural properties of objects in a mathematical system in one of two ways: (i) as properties *definable* from the primitive relations of a given system, or (ii) as properties of objects that are shared by *structurally similar* systems. Compare, for instance, Shapiro on the first approach in his account of *non-eliminative* structuralism:

> Define a property to be 'structural' if it can be defined in terms of the relations of a given structure. [Shapiro, 2008, p. 286][1]

The central idea here is that structural properties are precisely the properties that are *definable* in the language of a mathematical theory. The second approach, in contrast, is based on the notion of structural *invariance* or abstraction. Compare Linnebo on this account, again in the context of non-eliminative structuralism:

> A structural property can now be characterized as a property that can be arrived at through this process of abstraction, or, equivalently, a property that is shared by every system that instantiates the structure in question. [Linnebo, 2008, p. 64]

The main idea here is to specify structural properties of objects based on an act of abstraction from isomorphic systems: a property of objects in a system counts as structural if it also holds of all corresponding objects in isomorphic systems. So far, however, little work has been done to make these two approaches formally precise.[2]

In this paper, we aim to remedy this situation. We will present two formal explications of structural properties corresponding to the two informal characterizations above. We will call them the *definability account* and the *invariance account* of structural properties respectively. A central point to be made here is that each of these two accounts comes in two versions based on two ways of

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[1] It should be noted here that in Shapiro's account, the mathematical entities considered here are not concrete objects in model-theoretic systems, but 'places' in *ante rem* structures. Compare also [Shapiro, 2006] for a more detailed philosophical discussion of the notion of structural properties.

[2] An important exception is [Linnebo and Pettigrew, 2014] which contains a formal discussion of structure-abstraction principles for non-eliminative structuralism and of 'fundamental properties' of positions in such structures.

representing mathematical entities: (i) a version for structured mathematical *systems* and (ii) a version for entities conceived as *elements* in such systems. The reason for this bifurcation is that being a structural property means different things in those two contexts. Take the natural numbers as an example again. A natural-number system is a system of objects that satisfies the axioms of second-order Peano arithmetic. Such a system of objects may have additional structure, but in this context all we care about is the structure in virtue of which the system satisfies the theory in question. Intuitively, then, a structural property of a natural-number system is a property the system has or does not have in virtue of this structure.

The elements within a natural-number system, in contrast, are the numbers from the point of view of the system. A structural property of a number, then, is a property that the number has or does not have in virtue of the relevant structure of the system in which it occurs. The properties of a prime number are examples of structural properties in this sense, while the property of being a von Neumann ordinal is a counterexample. Thus, we have two senses of structural properties depending on two different contexts: if we look at structured systems as a whole, we get one sense of structural properties, and if we look at the elements within such structured systems, we get another sense.

Present work on mathematical structuralism focuses mainly on structural properties in the latter sense, that is, on the structural properties of elements in mathematical systems. This holds in particular for recent contributions to *non-eliminative* structuralism, specifically in the debate on the identity of structurally indiscernible places in pure structures. This discussion focuses on the adequacy of structuralist identity principles formulated in terms of structural properties of such places. The principle in question says that two places in a pure structure count as identical here if they share the same structural properties.[3] In turn, when philosophers discuss structural properties of mathematical systems — for instance in work on *eliminative* or *category-theoretic* structuralism — then usually no connection is made to these other debates.[4]

The present paper wants to bridge these different lines of research in philosophy of mathematics by giving a unified account of the notion. In particular, the first main goal here will be to show that both the invariance account and the definability account can be made to work in a precise sense for both types of mathematical properties, that is, for properties of systems as well as properties of elements in such systems. Our focus will consequently not be on a particular

[3] See, for instance, [Keränen, 2001; Shapiro, 2008; Ketland, 2006; Leitgeb and Ladyman, 2008]. Even though the debate centers around this and related principles of structural indiscernibility, structuralists do not universally accept the principle. Indeed, Leitgeb and Ladyman, among others, reject the principle as a criterion of identity for objects in (*ante rem*) structures.

[4] An important exception to this is [Landry and Marquis, 2005] where structured systems (such as groups, topological spaces, *etc.*) are treated as objects within category-structured systems. See [Awodey, 1996] for a general account of category-theoretic structuralism.

version of mathematical structuralism such as non-eliminative or eliminative structuralism. Neither will we take a stance on standard structuralist claims involving structural properties, such as the identity of structural indiscernibles or the so-called purity thesis, which states that mathematical objects have *only* structural properties (see [Linnebo and Pettigrew, 2014]). Rather, the aim will be to give a general conceptual and logical analysis of the notion that provides us a better understanding of how structural properties should be used in these philosophical debates.

The second main goal in the paper is to get a clearer understanding of the relation between the invariance-based and the definability-based accounts of structural properties. In particular, we will show that the two accounts do not characterize the same concept. Based on this observation, we propose a tolerant, Carnapian stance with respect to the choice of explication: we argue that neither of the two explications gives us the 'correct' notion of structural properties; instead both accounts have their philosophical and mathematical merits.

The paper will be organized as follows: In Section 2, we will lay the conceptual foundations for the rest of the paper. In particular, we will further explain the distinction between structural properties of systems of mathematical objects and structural properties of elements in such systems. In Section 3, we will present a generalized version of the invariance-based account of structural properties. Section 4 will present the explication of structural properties in terms of their definability in a mathematical language. Section 5 will then turn to a more general discussion of the conceptual relation between the two approaches. Specifically, we offer a philosophical assessment of the fact that the two accounts do not determine the same pre-theoretical notion of a structural property. Section 6 will contain a summary and some suggestions for future research.

## 2. MATHEMATICAL OBJECTS AND THEIR PROPERTIES

As mentioned in the introduction, structural properties will be specified here in accordance with two different ways to represent mathematical objects in particular contexts: (i) as elements in structured systems, and (ii) as the structured systems themselves. When we speak about mathematical entities in the latter sense, we usually refer to them in the context of certain axioms that describe the relevant structure of the system: systems are thus considered as *models* of these axioms. For instance, when the natural numbers are treated as a structured number system, this system is understood as a model of the axioms of second-order Peano arithmetic. Notice that mathematicians may talk differently about the same system in different contexts: for example, they may treat the natural numbers as a model of Peano arithmetic or as a monoid, *i.e.*, as a system satisfying the monoid axioms. Moreover, such systems can themselves be elements in larger mathematical systems: for example, the natural numbers viewed as a monoid are themselves elements in the system of submonoids of the natural numbers. Thus, how we conceptualize the mathematical objects we talk about is highly context sensitive.

In what follows, we will present both ways to think about mathematical entities in a standard model-theoretic setting. Structured systems always belong to a particular mathematical type, for instance the type of groups, graphs, or number systems. A specific type is usually defined by a set of axioms and formulated in an associated language. This is often a first-order language, for instance a first-order language describing abstract groups. Nevertheless, in the remainder of this paper, we will also consider higher-order languages for the description of mathematical systems and their properties. The second-order formulation of Peano arithmetic is a well-known case in point here. In the following account of mathematical types, systems, and objects, we deliberately leave the logical strength of the mathematical languages unspecified for the moment.

If $\mathbf{T}$ is a type of structured systems of mathematical objects, we denote the associated set of axioms by $\Lambda_{\mathbf{T}}$ and the language in which these axioms are formulated by $\mathcal{L}_{\mathbf{T}}$. Typically, the non-logical vocabulary of such a language contains a set of function symbols $\mathcal{F}$, a set of relation symbols $\mathcal{R}$, and a set of individual constants $\mathcal{C}$. For example, the vocabulary of a (first- or second-order) language $\mathcal{L}_{PA}$ of Peano arithmetic contains the function symbols $S, +$, and $\cdot$ for the successor function, addition, and multiplication, as well as the individual constant 0 for zero. One can view a structured system canonically as a model of $\mathcal{L}_{\mathbf{T}}$ which satisfies the axioms $\Lambda_{\mathbf{T}}$ — *i.e.*, we can view it as a model-theoretic system of the form

$$\mathcal{M} = \langle D^{\mathcal{M}}, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C} \rangle$$

that consists of a non-empty domain $D^{\mathcal{M}}$, a number of functions and relations over the domain that interpret the function and predicate symbols, as well as of a number of distinguished elements that interpret the individual constants of the language such that all of these components behave as $\Lambda_{\mathbf{T}}$ says. Given this, we can simply view elements of systems as *individuals* in the domain of a model-theoretic structure. So, if $N = \langle \mathbb{N}^N, 0^N, S^N \rangle$ is a natural-number system, then an element in $N$, *i.e.*, a natural number in the system, is simply an individual number $n \in \mathbb{N}^N$.

In this model-theoretic framework, the important notion of structural similarity between systems can be made precise in terms of the notion of isomorphism for models of the relevant language. This relation is defined in the usual way:

**Definition 1 (T-isomorphism).** *Two $\mathcal{L}_{\mathbf{T}}$-systems $\mathcal{M}$ and $\mathcal{N}$ are* isomorphic *if there exists a bijection $\lambda : D^{\mathcal{M}} \to D^{\mathcal{N}}$ such that:*

(i) $\lambda(c^{\mathcal{M}}) = c^{\mathcal{N}}$*, for all individual constants $c \in \mathcal{L}_{\mathbf{T}}$*
(ii) $(a_1, \ldots, a_n) \in R^{\mathcal{M}} \Leftrightarrow (\lambda(a_1), \ldots, \lambda(a_n)) \in R^{\mathcal{N}}$*, for all $n$-ary relation symbols $R \in \mathcal{L}_{\mathbf{T}}$ and $a_1, \ldots, a_n \in D^{\mathcal{M}}$.*
(iii) $\lambda(f^{\mathcal{M}}(a_1, \ldots, a_n)) = f^{\mathcal{N}}(\lambda(a_1), \ldots, \lambda(a_n))$*, for all $n$-ary function symbols $f \in \mathcal{L}_{\mathbf{T}}$ and $a_1, \ldots, a_n \in D^{\mathcal{M}}$.*

We say that two systems $\mathcal{M}$ and $\mathcal{N}$ of type $\mathbf{T}$ are isomorphic, in symbols $\mathcal{M} \simeq \mathcal{N}$, iff $\mathcal{M}$ and $\mathcal{N}$ are isomorphic as models of language $\mathcal{L}_{\mathbf{T}}$.

We next turn to the properties of mathematical objects. A lot can be said about the nature and existence of properties and philosophical positions on these matters abound.[5] We shall try to remain as neutral as possible on the subject. For the most part, an intuitive understanding of properties suffices: properties are simply those things that we can attribute to or predicate of things. Informally, one usually refers to properties using gerunds of the form 'being ...' or 'having ...'. For technical purposes, we distinguish between properties of elements in systems and properties of systems. For example, the property of being prime is a property of numbers in a number system and the property of having an infinite domain is a property of systems of a certain type. For both elements and systems, we write $\mathsf{P}(x)$ to indicate that the object $x$ has the property $\mathsf{P}$.

In the following, we will treat properties of mathematical systems as classes of such systems. For instance, the property of being a commutative group will be understood here as the class of Abelian groups.[6] In contrast, properties of elements in such systems will be understood (in a Lewisian sense) as functions from systems to sets of elements in these systems. For example, the property of being an even number will be treated as a function that maps to each number system satisfying second-order *PA* the set of even numbers in its domain.[7] Mathematical properties of both types can thus be expressed more formally in the following way:

**Definition 2 (Mathematical Properties).**

(1) *A property* $\mathsf{P}$ *of* $\mathbf{T}$*-systems is the class* $\{\mathcal{S} \in \mathbf{T} \mid \mathsf{P}(\mathcal{S})\}$ *of all and only the systems of type* $\mathbf{T}$ *that have the property* $\mathsf{P}$.

(2) *Let* $\mathsf{P}$ *be a property of elements of* $\mathbf{T}$*-systems. The local extension of* $\mathsf{P}$ *in a system* $\mathcal{S}$ *is the class* $\epsilon_{\mathcal{S}}\mathsf{P} = \{x \in \mathcal{D}^{\mathcal{S}} \mid \mathsf{P}(x)\}$ *of all and only those things in the domain of* $\mathcal{S}$ *that have the property. Property* $\mathsf{P}$ *is the function* $\iota\mathsf{P} : \mathcal{S} \mapsto \epsilon_{\mathcal{S}}\mathsf{P}$ *that assigns to every system* $\mathcal{S}$ *of type* $\mathbf{T}$ *the local extension* $\epsilon_{\mathcal{S}}\mathsf{P}$ *of* $\mathsf{P}$ *in system* $\mathcal{S}$*.*[8]

Given this account, three points should be noted here. First, since most types of mathematical systems (*e.g.*, the type of Abelian groups) are not sets but,

[5] For an overview see, for instance, [Oliver, 1996].

[6] Strictly speaking, extensions of mathematical properties so construed are often *proper classes*. An axiomatic class theory such as von Neumann-Bernays-Gödel set theory (NBG) would therefore be a suitable theoretical framework to express the present account of mathematical properties more formally.

[7] This functional account of mathematical properties essentially conforms with Lewis's [1986] possible-worlds approach to properties.

[8] In the remainder of the paper, we will also use the following notation to speak about the local extensions of properties of elements in particular systems: if $\mathcal{M}$ is a mathematical system, $d \in D$ an object in this system, and $\mathsf{P}$ a property, then we also write $\mathsf{P}_{\mathcal{M}}(d)$ to say that $d$ is in the extension of $\mathsf{P}$ in $\mathcal{M}$.

strictly speaking, proper classes, it follows that the mathematical properties of elements in such systems, if understood in the above sense, cannot be presented by set-theoretic functions. Instead, they have to be thought of as proper-class-sized functions, *i.e.*, as functions between proper classes.[9]

Second, given the present account, one can think of a mathematical property of elements as being instantiated in different systems of a given type. Consider again of the property of being even in the context of number systems: this number-theoretic property is understood here as a function from systems satisfying second-order *PA* to particular number sets, namely the sets of even numbers in the particular model considered. As will be shown in the next section, these local extensions of the property can differ from system to system. For instance, given two different set-theoretic models of *PA*, the one based on the von Neumann construction of the natural numbers and the other on Zermelo's construction of the natural numbers, the respective sets of even numbers will clearly be distinct. Nevertheless, the present account allows us to treat these local extensions as belonging to the general arithmetical property that can be instantiated in all of these systems. This functional understanding of properties of elements is clearly motivated by a structuralist account of mathematics. In particular, a structuralist would think of 'being even' as a general number-theoretic property that is independent of any particular system satisfying the theory.[10]

Finally, following Quine's famous dictum of 'no entity without identity', the present account of mathematical properties also calls for a specification of their identity conditions. When should we be committed to saying that two mathematical properties are identical? The philosophical literature on properties discusses a number of possible criteria of property individuation suitable for this task. Generally speaking, one can think of these identity criteria as laws of the following form:

For all properties $\mathsf{P}$ and $\mathsf{Q}$: $\mathsf{P} = \mathsf{Q}$ iff $\mathcal{C}(\mathsf{P}, \mathsf{Q})$,

where $\mathcal{C}(\mathsf{P}, \mathsf{Q})$ is a condition, usually expressed in the form of an equivalence relation between the properties $\mathsf{P}$ and $\mathsf{Q}$. If we restrict our attention to the case

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[9] The notion of classes is treated informally in standard set theory; that is, classes are not described by the axioms of (first-order) *ZF*, but treated as definable predicates in the language of set theory. A proper-class-sized function $F$ can also be dealt with in *ZF* by representing it as a first-order definable formula of the form $\Phi(x, y)$ such that $\Phi(x, y)$ holds in $V$ if and only if $F(x) = y$, where by $V$ we mean the cumulative von Neumann hierarchy of sets. Proper-class-sized functions can also be characterized more explicitly in an axiomatic class theory such as NBG or Morse-Kelley set theory. Compare [Jech, 2002, pp. 5–6] for details. We would like to thank an anonymous reviewer for emphasizing this point in his report.

[10] From the perspective of non-eliminative structuralism, it is tempting to say that such general mathematical properties hold primarily of pure positions in abstract structures. Since these structures can be exemplified by more concrete systems, it follows that the properties also apply to elements in systems that instantiate these pure positions. In the present paper, we choose to treat mathematical properties of elements functionally instead of adopting such a non-eliminative account of structures.

of properties of elements in systems, one natural approach would be to specify identity in terms of co-extensionality in all systems:

> For all properties $\mathsf{P}$ and $\mathsf{Q}$ of objects in systems of type $\mathbf{T}$: $\mathsf{P} = \mathsf{Q}$ iff for all systems $\mathcal{S}$ of $\mathbf{T}$ and for all objects $x \in \mathcal{D}^{\mathcal{S}}$: $\{x \in \mathcal{D}^{\mathcal{S}} \mid \mathsf{P}(x)\} = \{x \in \mathcal{D}^{\mathcal{S}} \mid \mathsf{Q}(x)\}$.

For most applications discussed in the following, this identity criterion will be adequate. However, one might argue that it is still too coarse-grained for the individuation of mathematical properties. In particular, given the functional account of properties outlined above, one will be forced to identify properties that are intuitively distinct. Consider again of the case of arithmetic: the properties of 'being the square of 2' and of 'being the fourth successor of 0' will turn out as identical according to the above criterion since they share the same extension in every system satisfying *PA*.

Given these reasons, one might consider other, so-called *hyperintensional* identity criteria for mathematical properties. These criteria are finer-grained than co-extensiveness in all systems. Recent work on the metaphysics of properties has focused on different versions of *structured*-property theories. Very roughly, properties so conceived are either primitive or equipped with some internal propositional structure. The identity of complex properties is then determined by reference to this internal structural composition.[11] We will not be able to discuss such hyperintensional identity criteria and their possible relevance in the context of mathematical properties any further here. Instead, we simply acknowledge the fact that thinking about identity of mathematical properties in terms of co-extensiveness can lead to problematic results. A more definitive theory of identity conditions for mathematical properties will have to be developed elsewhere.

Given this general model-theoretic picture of mathematical systems, objects, and their properties, when do mathematical properties qualify as structural? Intuitively, a structural property is a property a mathematical object has *in virtue* of or *because of* its structure. As should be clear, this means different things for systems and elements of such systems: a structural property of a system is a property the system has because of its internal structure — it tells us something about the structural composition of the system. In the case of elements in structured systems, in turn, structural properties are properties that express information about the role of the elements in the overall structure of the system. Put differently, these are properties a particular element has because of its *contextual* structure, *i.e.*, the relations in which it stands with the other

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[11] See, for instance, [Menzel, 1993] for an algebraic approach and [King, 2007] for a 'quasi-syntactic' approach to structured properties. The conception of quasi-syntactically structured properties in the context of mathematical properties can already be found in [Lewis, 1986, pp. 56–58].

elements of the system it belongs to.[12] The aim in the next two sections will be to see how these two informal ways of thinking about structural properties can be made formally precise in terms of the notions of invariance and definability.

## 3. THE INVARIANCE ACCOUNT

One way to specify structural properties is based on the notion of *invariance* under structure-preserving transformations. This notion has a long mathematical history, tracing back to nineteenth-century work in algebra and algebraic geometry. The first attempt to define the notion of structural properties *explicitly* in terms of invariance can be found in Carnap's early work on axiomatics. In his manuscript *Untersuchungen zur allgemeinen Axiomatik*, Carnap gives us the following definition:

> **Definition 1.7.1** The property $fP$ of relations is called a 'structural property' if, in case it applies to a relation $P$, it also applies to any other relation isomorphic to $P$. To say that $fP$ is a structural property is expressed in a formula as follows: $(P, Q)[(fP \& Ism(Q, P)) \rightarrow fQ]$. The structural properties are so to speak the invariants under isomorphic transformation. They are of central importance for axiomatics. [Carnap, 2000, p. 74]

Carnap defines structural properties of relations here as those properties that remain invariant under isomorphisms. How can we *generalize* this invariance-based approach to apply to properties of arbitrary mathematical objects? To address this, it should be emphasized that something Carnap's account does *not* give us is an understanding of structural properties in terms of more basic or non-structure-related notions. His definition can be taken to rely in one way or another on some *primitive* structural facts that determine whether two objects count as structurally equivalent or isomorphic. To see this, consider again how the relation of isomorphism for a given class of mathematical objects is usually defined. The 'traditional way' specified in Section 2 — and already anticipated by Carnap — is by first identifying a set of primitive properties and relations and then to defining two objects as being isomorphic if and only if there is a bijection between them that preserves these primitive properties.[13]

Carnap's definition is thus based conceptually on a prior axiomatic stipulation of certain facts that are taken as our basis for speaking about structure and structural invariance. What the above definition *does* is that it extends these axiomatically specified properties to arbitrary properties. This said, it is

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[12] Examples of such properties can be traced back to Benacerraf's famous paper 'What number could not be' from 1965: 'Being prime' or 'being even' are structural properties of individual natural numbers whereas 'being a specific Zermelo number' is not.

[13] Take groups as a concrete example. A *group isomorphism* between groups $G$ and $G'$ is a bijective function from $\mathbb{G}$ to $\mathbb{G}'$, such that $g \circ^G h = j$ iff $f(g) \circ^{G'} f(h) = f(j)$. Thus, the relevant primitive property is having a group operation $\circ$. A group isomorphism is a bijection that preserves this property.

explicitly left open how we identify the primitive properties of mathematical objects in the first place. Moreover, these properties clearly differ for different classes of mathematical objects. For instance, an object $N$ is isomorphic to the natural numbers $\mathbb{N}$ iff there is a bijection from $\mathbb{N}$ to $N$ that preserves the successor function and the distinguished object $0 \in \mathbb{N}$. Similarly, an object $R$ is isomorphic to the reals $\mathbb{R}$ iff there is a bijection from $\mathbb{R}$ to $R$ that preserves the distinguished objects $0 \in \mathbb{R}$ and $1 \in \mathbb{R}$, the addition function $+$, the multiplication function $\cdot$, and the order $\leq$ of the reals. Thus, for different kinds of mathematical objects we have different kinds of axiomatically defined constitutive properties.

Based on these considerations, we can outline a modernized account of Carnap's definition of structural properties. Specifically, we will assume that a definition has to be relativized to a particular type of mathematical objects. As mentioned above, this type comes with a distinguished 'structural vocabulary', *i.e.*, a set of primitive terms in the formal language of the theory. The explication of structural properties of structured objects or systems of a given mathematical type then looks as follows:

**Explication 1.** *A property* $\mathsf{P}$ *is a* structural property *of systems of type* $\mathbf{T}$ *iff*

$$\forall \mathcal{S}, \mathcal{S}' \in \mathbf{T} : \mathsf{P}(\mathcal{S}) \ \& \ \mathcal{S} \simeq \mathcal{S}' \Rightarrow \mathsf{P}(\mathcal{S}').$$

Structural properties of systems of a given type are thus properties that remain invariant under the isomorphisms between systems of that type. For instance, being a graph of a certain *order* or *clique number* clearly turns out as a structural property of graphs under this condition. Similarly, being an infinite cyclic group turns out as a structural property of groups.[14]

Our discussion of the invariance-based account has so far focused on properties of systems, that is, mathematical objects that already possess some kind of internal structural composition. What about properties of elements in such a system? Interestingly, several analogous definitions of structural properties of individuals in a structured system can be given in terms of invariance conditions. One possible approach here is to restrict attention to the structure-preserving permutations of a given domain in which such objects occur. Let $\mathcal{A}$ be a particular model-theoretic system. The objects considered now are the elements in $D$. Properties of such elements can be treated extensionally as subsets of the system's domain. Given this, a possible invariance condition for such properties is based on the automorphisms of $\mathcal{A}$, that is, isomorphisms of the form $f : D \to D$. Specifically, we can say that a property of elements in

[14] An enticing idea in this context would be simply to consider structures as limit cases of structural properties of mathematical systems. For instance, we might understand *the* natural-number structure as the structural property of being isomorphic to the canonical natural-number system $\mathbb{N}$. We would like to thank a referee for pointing us to this possibility.

$\mathcal{A}$ is structural if and only if it remains invariant under every automorphism on $\mathcal{A}$.[15]

The symmetry between this approach and Explication 1 should be clear enough: in both versions, the 'structural' character of a property is defined in terms of its invariance under certain structure-preserving transformations. In the first case, this is invariance under isomorphisms, in the second case it is invariance under automorphisms, *i.e.*, the inner isomorphisms of a given system. In spite of this nice symmetry, the automorphism-based account does *not* give us a materially adequate account of structural properties of objects in a given system. One can easily construct counterexamples, namely properties that turn out to be structural in this sense but fail to be structural from a pre-theoretical understanding of the term. Typical cases in point here are properties of elements in *rigid* systems, *i.e.*, systems without non-trivial automorphisms. Examples of this are the natural-number systems satisfying second-order *PA* or the real-number field. Given that the only automorphisms on these systems are the respective identity mappings, it follows that every property of elements in their domains is by definition invariant in the above sense. This includes properties such as 'being someone's favorite numbers' which clearly fail to count as structural from an intuitive point of view.[16]

Fortunately, there exists an alternative explication of structural properties of elements in mathematical systems that is also in direct symmetry with Explication 1. The underlying idea here is to specify the notion not in terms of invariance under automorphisms of a single given system, but — as in the above case — in terms of the invariance under isomorphisms between different systems:

**Explication 2.** *Let $\mathcal{S}$ be a system of type* $\mathbf{T}$. *Then a property* $\mathsf{P}$ *is a structural property of the objects in the domain of $\mathcal{S}$ iff for all systems $\mathcal{S}'$ (also of type $\mathbf{T}$) and for all isomorphisms $\lambda : D^{\mathcal{S}} \rightarrow D^{\mathcal{S}'}$:*

$$\forall x \in D^{\mathcal{S}} : \mathsf{P}_{\mathcal{S}}(x) \Rightarrow \mathsf{P}_{\mathcal{S}'}(\lambda(x))$$

Structural properties of objects in a system $\mathcal{S}$ are specified here as those properties that the objects 'keep' when making isomorphic copies of $\mathcal{S}$.[17] Consider, for instance, the property of having no predecessor in the context of systems

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[15] More formally, we say that a property $\mathsf{P}$ of objects in $\mathcal{A}$ is a *structural property* iff it is invariant *in* $\mathcal{A}$, *i.e.*, for every element in the automorphism class $f \in Aut(\mathcal{A})$, we have $f(\mathsf{P}) = \mathsf{P}$. The idea of characterizing structural properties in terms of invariance under automorphism has also been discussed in non-eliminative mathematical structuralism, in particular, in [Keränen, 2001].

[16] It is interesting to see that the distinction between rigid and non-rigid systems also plays a key role in recent work on non-eliminative structuralism, in particular, on the identity of indiscernible positions in a pure structure. See, *e.g.*, [Keränen, 2001; Leitgeb and Ladyman, 2008; Shapiro, 2008].

[17] The notion of structural properties is defined here as a binary relation between systems and properties of elements in these systems. It could also be defined as a ternary

satisfying *PA*. In the standard system of natural numbers, this property is true only of the number zero. Furthermore, the property turns out as structural in the above sense since it also applies to all isomorphic copies of zero, *i.e.*, to the 'base element' in any other model of *PA*.

To illustrate this isomorphism-based account further, let us look at another example. Take the arithmetical property of being an even number discussed already in the previous section. For present purposes, we will focus on two particular number systems of *PA*. The first one can be called the von Neumann system: natural numbers are represented here by sets in the following way: $0^{\mathrm{vN}} = \emptyset, 1^{\mathrm{vN}} = \{\emptyset\}, 2^{\mathrm{vN}} = \{\emptyset, \{\emptyset\}\}, \ldots$, and $\mathbb{N}^{\mathrm{vN}} = \{0^{\mathrm{vN}}, 1^{\mathrm{vN}}, \ldots\}$. The successor function is specified as $S^{\mathrm{vN}}(n) = n \cup \{n\}$, for any $n \in \mathbb{N}^{\mathrm{vN}}$. The system $N^{\mathrm{vN}} = \langle \mathbb{N}^{\mathrm{vN}}, 0^{\mathrm{vN}}, S^{\mathrm{vN}} \rangle$ satisfies second-order *PA*. The second number system is the Zermelo system: natural numbers are identified here with sets in the following way: $0^{\mathrm{Z}} = \emptyset, 1^{\mathrm{Z}} = \{\emptyset\}, 2^{\mathrm{Z}} = \{\{\emptyset\}\}, \ldots$, and $\mathbb{N}^{\mathrm{Z}} = \{0^{\mathrm{Z}}, 1^{\mathrm{Z}}, \ldots\}$. The successor function is specified as $S^{\mathrm{Z}}(n) = \{n\}$, for any $n \in \mathbb{N}^{\mathrm{Z}}$. The system $N^{\mathrm{Z}} = \langle \mathbb{N}^{\mathrm{Z}}, 0^{\mathrm{Z}}, S^{\mathrm{Z}} \rangle$ is also a model of second-order *PA*. Benacerraf's central insight in [Benacerraf, 1965] was that, from a purely structural point of view, it does not matter which of the two set-theoretic systems is taken to represent the natural-number structure. Neither of them should therefore be singled out as the preferred model of arithmetic. Put differently, both systems are suited equally well for this task since the elements in them share the same structural, *i.e.*, purely relational properties. Based on this insight, we suggested treating arithmetical properties such as 'being an even number' as functions from *PA*-systems to subsets of elements in them. In the particular example, the function presenting the property of being even will pick out the set $\{2^{\mathrm{vN}}, 4^{\mathrm{vN}}, 6^{\mathrm{vN}}, \ldots\}$ relative to the von Neumann system, the set $\{2^{\mathrm{Z}}, 4^{\mathrm{Z}}, 6^{\mathrm{Z}}, \ldots\}$ relative to the Zermelo system, and corresponding sets for any other model of *PA*. This property is clearly structural according to Explication 2: systems $N^{\mathrm{vN}}$ and $N^{\mathrm{Z}}$ are isomorphic and any isomorphism between them maps the object $2^{\mathrm{vN}}$ to object $2^{\mathrm{Z}}$, object $4^{\mathrm{vN}}$ to $4^{\mathrm{Z}}$, object $6^{\mathrm{vN}}$ to $6^{\mathrm{Z}}$, and so on. Thus, the property of being even is preserved under isomorphisms in the sense specified above.

Does Explication 2 rule out the kind of counterexamples to the automorphism-based approach mentioned above? Consider again the case of accidental properties such as being someone's favorite numbers. How such properties are to be evaluated depends on how they are understood given the present framework. In our view, the most natural way to interpret them is to say that such properties apply to objects of a particular number system, for instance, of a particular model of *PA*. Thus, we take it that such properties are about a particular set of natural numbers in a particular number system. To give an example, consider the property 'being one of Zermelo's favorite numbers' (henceforth $\mathsf{Zer}$) and let Zermelo's favorite numbers be the numbers in system $N^{\mathrm{Z}}$ which form his birth date $\{1871^{\mathrm{Z}}, 7^{\mathrm{Z}}, 27^{\mathrm{Z}}\}$. As can easily be shown, $\mathsf{Zer}$

relation between systems, properties, and particular elements in the following way: a property $\mathsf{P}$ is a structural property of an object $a$ in system $\mathcal{S}$ iff for all systems $\mathcal{S}'$ (also of type $\mathbf{T}$) and for all isomorphisms $\lambda : D \to D'$: $\mathsf{P}_{\mathcal{S}}(a) \Rightarrow \mathsf{P}_{\mathcal{S}'}(\lambda(a))$.

turns out to be invariant and thus structural (relative to $N^{\mathrm{Z}}$) according to the automorphism-based approach. However, it fails to be structural according to the isomorphism-based account. One way to think of Zer more formally is as a function that gives us the set $\{1871^{\mathrm{Z}}, 7^{\mathrm{Z}}, 27^{\mathrm{Z}}\}$ for system $N^{\mathrm{Z}}$ and the empty set for any other *PA*-system. This treatment reflects the fact the Zer applies only to certain elements of system $N^{\mathrm{Z}}$ and to no objects in any other number system. As a consequence, Zer is not preserved by isomorphisms between *PA*-systems and thus fails to be structural. In this respect, the present approach is clearly preferable to the automorphism-based account.[18]

Note that Explication 2 of the notion of structural properties presupposes the functional understanding of mathematical properties of elements in systems outlined in Section 2. Thus, in contrast to the automorphism-based approach, it makes little sense to conceive of properties purely locally as sets of elements in a particular system. Rather, we have to assume that a property can recur in different mathematical systems and have different interpretations (or local extensions) in them. It is therefore natural to think of properties as functions from systems to such local extensions, *viz.*, as sets of individuals in a given system, in the sense specified in Section 2. Viewed in this way, a property qualifies as structural if there exists, between its local extensions in any two systems, a bijective correlation that is induced by an isomorphism between the systems.[19]

This functional treatment of mathematical properties (of elements in systems) also allows us to address a possible objection regarding the descriptive adequacy of Explication 2.[20] Consider again the example of 'being one of Zermelo's favorite numbers' but let this now be the property of being a prime number in a *PA*-system. Is this property structural? This again depends on whether the property is interpreted only locally, *i.e.*, as applying exclusively to

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[18] It should be noted here that there exists a second way to interpret 'being one of Zermelo's favorite numbers'. According to this view, the property applies not to elements of a particular *PA*-system, but rather to abstract number positions that can be exemplified by such set-theoretic objects. In particular, given the present framework, we could think of Zer as a function that assigns the respective interpretations of the numerals '1871', '7', and '27' for each *PA*-system. Understood in this way, Zer is a structural property.

[19] How are the two explications of structural properties suggested here related to the different versions of mathematical structuralism discussed in the literature? The automorphism-based approach sides well with non-eliminative structuralism and its focus on properties of *positions* in pure structures. In contrast, Explication 2 seems closer in spirit to eliminative structuralism given that structural properties are specified here for objects in systems and in terms of the generalization over systems and isomorphisms between them. That said, it is interesting to compare Explication 2 with the treatment of structural properties in the context of non-eliminative structuralism given in [Linnebo and Pettigrew, 2014]. Linnebo and Pettigrew specify pure structures as well as the positions and relations in such structures in terms of *Fregean* abstraction principles. 'Fundamental relations' of such positions are constructed by abstraction from relations on the domain of a concrete system. Linnebo and Pettigrew further show that fundamental properties so construed are structural in the sense specified in Explication 2.

[20] We would like to thank one of the anonymous reviewers for pointing out this objection to us and for very helpful suggestions.

objects of a particular system or more generally, as a property applying to the prime numbers in all *PA*-systems. Considering the first — in our view more natural — interpretation, one could say that the property of being Zermelo's favorite numbers (henceforth $\mathsf{Zer}^*$) is given by the set $\{2^{\mathrm{Z}}, 3^{\mathrm{Z}}, 5^{\mathrm{Z}}, 7^{\mathrm{Z}}, 11^{\mathrm{Z}}, \dots\}$ of objects in the Zermelo system. Alternatively, we could understand $\mathsf{Zer}^*$ as the function that selects this set relative to system $N^{\mathrm{Z}}$ and the empty set for any other *PA*-system. So construed, the property is not isomorphism invariant and thus not structural.

Considering the second option, one could also understand 'being one of Zermelo's favorite numbers' more generally as the property of prime numbers in all *PA* systems (henceforth $\mathsf{Zer}^{\ddagger}$). Thus, again adopting our functional treatment of properties of elements, the property can then be viewed as the function that selects the set of prime numbers in each *PA* system. As mentioned above, accidental properties of elements in systems of a given type (such as the property of belonging to person X's favorite numbers) should intuitively not count as structural, simply because they are not about the internal structure of the systems in question. However, $\mathsf{Zer}^{\ddagger}$ clearly turns out to be structural according to the isomorphism-based account since it is invariant under any isomorphic transformation of the standard natural-number system.[21]

Does the property 'being one of Zermelo's favorite numbers' present a counterexample to Explication 2? In the local version of it, *i.e.*, understood as $\mathsf{Zer}^*$, the property clearly presents no problem to the isomorphism-based account. In the general version $\mathsf{Zer}^{\ddagger}$, the situation is less obvious. In particular, an answer to the question depends on the choice of a particular criterion of identity that one wants to adopt for mathematical properties. Recall from Section 2 that different criteria can be considered here. As we saw, one natural approach is to take two properties to be identical if they are co-extensive in all possible systems considered. Applied to the present example, this would imply that the number-theoretic properties 'being one of Zermelo's favorite numbers' (interpreted as $\mathsf{Zer}^{\ddagger}$) and 'being a prime number' are in fact identical. Following a reviewer's suggestion, we might say that being one of Zermelo's favorite numbers (understood in this way) *is* in fact the property of being a prime presented in an exotic disguise. Moreover, as we saw, it also turns out to be a structural property, at least if one adopts Explication 2 as a way to precisify this notion. Therefore, at least if identity of properties is captured in terms of co-extensiveness in all systems, property $\mathsf{Zer}^{\ddagger}$ does not present a counterexample to Explication 2. A different assessment may be needed, however, if other, finer-grained criteria of identity for properties are considered. A more detailed discussion of the intricate relationship between different identity criteria for mathematical properties and the notion of structural properties (as specified in Explication 2) would

[21] Consider again the two systems of *PA*, namely the Zermelo system and the von Neumann system. Relative to these two systems, property $\mathsf{Zer}^{\ddagger}$ picks out two infinite sets, namely $\{2^{\mathrm{Z}}, 3^{\mathrm{Z}}, 5^{\mathrm{Z}}, 7^{\mathrm{Z}}, 11^{\mathrm{Z}}, \dots\}$ and $\{2^{\mathrm{vN}}, 3^{\mathrm{vN}}, 5^{\mathrm{vN}}, 7^{\mathrm{vN}}, 11^{\mathrm{vN}}, \dots\}$ respectively. Since any isomorphism between $N^{\mathrm{Z}}$ and $N^{\mathrm{vN}}$ will preserve this property, it turns out to be structural.

go beyond the scope of this paper. We therefore leave this issue for possible future work.

Given the two invariance-based explications of structural properties, two further points of commentary are in order here. Notice first that in contrast to Carnap's original account, the modernized versions only give a *partial* definition of structural properties relative to a given mathematical type or to particular objects of systems of a given type. Moreover, the definitions are *non-reductive* in the sense that they assume a prior specification of the primitive structural terminology used to describe the objects of this type. Thus, the notion of structure-preserving mappings is presupposed in the invariance-based account. As a consequence of this, the approach is sensitive to how we present objects and systems, that is what types and languages we associate with them.

Second, as already indicated above, both accounts are materially adequate in the sense that they agree with many of our intuitions about what counts as a structural property. In particular, the standard examples of such properties mentioned in [Benacerraf, 1965] all turn out to be structural according to Explications 1 and 2: having infinitely many prime numbers is a structural property of *PA*-systems. Being prime is a structural property of numbers in such systems. In contrast, having the set of von Neumann ordinals as the domain is *not* a structural property of *PA*-systems. Being a specific set is also *not* a structural property of numbers in such systems. Thus, both invariance-based explications seem to reflect closely our pre-theoretical understanding of structural properties in mathematics.

## 4. THE DEFINABILITY ACCOUNT

The second approach to defining structural properties is based on the notion of *logical definability*. As with the invariance-based approach, it has a long history, tracing back to early work on formal philosophy and the logic of science. In Russell's monograph *The Analysis of Matter*, one can find the view that a relation's 'structure is what can be expressed by mathematical logic' [Russell, 1927, p. 254]. The same view is elaborated more fully in Carnap's *Der Logische Aufbau der Welt* [1928]. The position defended there is that the structure of a relation is presented by 'the totality of its formal properties'. Formal or structural properties in turn are specified in the following way:

> By formal properties of a relation, we mean those that can be formulated without reference to the meaning of the relation and the type of objects between which it holds. They are the subject of the theory of relations. The formal properties of relations can be defined exclusively with the aid of logistic symbols, *i.e.*, ultimately with the aid of the few fundamental symbols which form the basis of logistics (symbolic logic). [Carnap, 1928, p. 21]

Thus, unlike in [Carnap, 2000], structural properties are not defined here in terms of invariance under isomorphisms, but in terms of their logical definability. How can we make this second approach more precise? As in the above case, it makes sense to highlight some facts about Carnap's account that need further consideration.

First, properties are defined here not only for mathematical relations, but for *all* possible relations, including physical or empirical ones. In the present context, the approach of specifying structural properties in terms of their definability can be restricted to the case of mathematical properties. The second point concerns what is meant by 'logical' in talk about logical definability here. In Carnap's case, definability in logic means expressibility in a *pure* logical language, *i.e.*, a language without non-logical vocabulary. Thus, according to this account, structural properties are precisely those properties expressible in pure higher-order logic. In the context of mathematics, this approach seems insufficient. When mathematicians speak of definable properties here, they usually have in mind formal languages that additionally contain some primitive mathematical vocabulary. Any modern reconstruction of the present account will thus have to be explicit about *both* the logical resources of a given language and its mathematical signature.

The final point concerns the metatheoretic notion of 'definability' in use here. For Carnap, being definable means that a property can be *explicitly* defined in the background language. Such definitions are conceived of purely syntactically by him, *i.e.*, as expressions 'formulated without reference to the meaning' of the relations and their relata [Carnap, 1928, p. 21]. In contrast to this, our present account of mathematical properties will be based on a model-theoretic understanding of definability.[22]

With these points in mind, we can turn to an explication of structural properties in terms of logical definability. We say that structural properties of systems can be specified in terms of the notion of definable *classes* of models of type $\mathbf{T}$ in the following sense:

**Explication 3.** *A property* $\mathsf{P}$ *is a structural property of systems of type* $\mathbf{T}$ *iff there is a closed formula* $\varphi \in \mathcal{L}_{\mathbf{T}}$ *such that* $\varphi$ *defines* $\mathsf{P}$*,* i.e.:

$$\{\mathcal{S} \in \mathbf{T} \mid \mathsf{P}(\mathcal{S})\} = \{\mathcal{S} \in \mathbf{T} \mid \mathcal{S} \vDash \varphi\}.$$

Structural properties of systems in this account are properties whose extensions — conceived here as classes of models of a given type — are definable in the associated language of that type. Consider some examples of mathematical properties that turn out to be structural in the above sense: the property of a group $G$ to have $\kappa$ elements in the underlying set for $\kappa$ a finite cardinal number is definable in the language of groups. The property of a graph to have $n$ edges is definable in the language of graph theory.

This explication applies to properties of mathematical systems with some kind of internal structural composition. As in the case of the invariance

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[22] See, *e.g.*, [Marker, 2000, §1.3] for a detailed discussion of the notion of definability in model theory.

approach, an analogous explication can be given for properties of elements in such systems. One possible way to go here is to specify structural properties relative to a single system. In this case, $\mathcal{L}_{\mathbf{T}}$-definability does not concern classes of models of a certain signature, but sets or relations *in* such a given system. Structural properties of elements in a system $\mathcal{S}$ of type $\mathbf{T}$ are thus properties whose extensions in $\mathcal{S}$ are definable in the associated language $\mathcal{L}_{\mathbf{T}}$. This account corresponds closely to the automorphism-based approach stated in Section 3. As we saw, invariant properties of objects can also be specified relative to a particular system, namely in terms of the invariance under all automorphisms of that system.

We have mentioned, however, that these 'local' approaches are not fully satisfactory for present purposes. In particular, they fail to capture our general 'structuralist' motivation for these definitions, namely to describe properties as entities that apply to individual objects *across* systems.[23] Thus, according to this view, the property of 'being prime' is not considered here as a property of natural numbers of a particular number system, but as an arithmetical property that applies to objects in all systems satisfying *PA*. For this reason, we propose a more general explication. First, we make sure that whenever we have a system $\mathcal{S}$ of type $\mathbf{T}$, we have a name for every member of the domain $D$ of $\mathcal{S}$. We achieve this by adding to the language $\mathcal{L}_{\mathbf{T}}$ an individual constant $\underline{d}$ for every member $d \in D$. This gives us the extended language $\mathcal{L}^{+}_{\mathbf{T}}$. We then move to the extended system $\mathcal{S}^{+}$, which is defined just like $\mathcal{S}$, except that we stipulate that $\underline{d}^{\mathcal{S}^{+}} = d$. Then we say:

**Explication 4.** *A property* $\mathsf{P}$ *is a structural property of elements in* $\mathbf{T}$*-systems iff for every system* $\mathcal{S}$ *of type* $\mathbf{T}$ *there is a formula* $\varphi(x) \in \mathcal{L}_{\mathbf{T}}$ *such that*:

$$\{d \in D \mid \mathsf{P}_{\mathcal{S}}(d)\} = \{d \in D \mid \mathcal{S}^{+} \vDash \varphi(\underline{d})\}.$$

According to this account, structural properties of elements in the systems of a given mathematical type are thus properties whose local extension in each system is definable in the associated language.

Some points of commentary are again in order here. First, notice that both definability-based explications closely capture our informal ways of thinking about structural properties (as outlined in Section 2). Recall that a structural property of elements in a system is usually understood as a property expressing some piece of information about their *relational* or *contextual* structure, *i.e.*, about the relations in which these objects stand to other objects in the system.

[23] As mentioned before, such local approaches to invariance and definability are compatible with a structuralist account of mathematics if one adopts a non-eliminative understanding of structures. Structures are then conceived as universals or patterns with a domain of abstract positions that can be instantiated by objects of concrete mathematical systems. Definable properties in this context do not concern objects in systems but rather their abstract placeholders in such structures. See, *e.g.*, [Keränen, 2001].

Exactly this understanding of structural properties is made precise in Explication 4. Similarly, we can say that the informal understanding of structural properties of structured systems is captured closely by Explication 3: structural properties are conceived here as those properties expressing a fact about the structural composition of the systems considered. Here again, definability in terms of the primitive terminology of a mathematical theory secures that the properties so expressed are about these *internal* or *intrinsic* structural facts.[24]

Second, it is insightful to see how the present approach differs *conceptually* from the invariance-based approach outlined above. Notice that Explications 3 and 4 also give us a *partial* and *non-reductive* account of structural properties. The account is partial because structurality is always specified relative to a particular mathematical context or type. It is non-reductive in the sense that a prior identification of primitive terminology is assumed that allows us to identify what we mean by structure in this particular context in the first place. Moreover, just as in the invariance approach, these explications are sensitive to how we represent objects and systems, that is how we set up the languages to describe them.

The central difference from the invariance approach is that the present account is also highly sensitive to the expressive power or logical strength of the associated language. Thus, it makes a difference whether the mathematical language $\mathcal{L}_{\mathbf{T}}$ in use is first-order or higher-order. To give just one example: some fairly simple graph properties — for instance, properties concerning the clique number of a graph — are not expressible in a first-order framework and are thus not structural according to Explication 3. They do become definable and thus structural, however, if one adopts a second-order language that allows one to quantify over subsets of a graph's vertex set. More generally, we can say that what counts as a structural property according to the definability approach is strongly dependent on the choice of the background language. It depends both on the mathematical (*viz.*, non-logical) signature as well as the logical resources (*viz.*, the types of variables and types of quantifiers that are part of the language) in use.

## 5. COMPARISON

Two different explications of the notion of structural properties were presented here, both of which seem to capture our pre-theoretical understanding of such properties in mathematics. In light of this, a natural question to ask is whether the invariance-based and the definability-based accounts are in fact equivalent. Put differently, do they determine the same collection of properties for a given type of mathematical objects? *Prima facie*, this seems a plausible assumption to make. The symmetry between invariance and definability has long been investigated in model theory. Thus, the approaches can be considered as two sides of

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[24] This does not mean that the definitions expressible in a formal language necessarily capture *all* structural facts about the mathematical systems considered. As will be pointed out below, the logical strength of a language in use also plays a significant role here.

the same coin, that is, as two ways of describing the structure of mathematical objects.[25]

This fact has also been acknowledged in the more recent literature on mathematical structuralism. For instance, a close variant of the definability-based explication of structural properties of objects in a system is discussed at length in the context of non-eliminative structuralism in [Keränen, 2001]. Keränen introduces there the notion of 'inter-structural' relational properties of *places* in a pure structure that can be instantiated by the elements of concrete systems. These properties are, he points out, definable in the relevant mathematical language (without individual constants) [2001, pp. 315–317]. Keränen goes on to argue that

> ... a property is *guaranteed* to be invariant under the automorphisms of **S** if and only if it can be specified by formulae in one free variable and without individual constants. [2001, p. 318]

Unfortunately, the relation between the invariance-based and the definability-based accounts is not as clear as is suggested there. In fact, Keränen's observation does not hold in general. It is not always the case that the explications of structural properties in terms of invariance match those given in terms of definability in a formal language. To see this, a simple cardinality-based argument can be given. Consider again the standard system of natural numbers $N = \langle \mathbb{N}^N, 0^N, S^N \rangle$ (which, for the sake of concreteness, we may take either to be $N^{\mathrm{Z}}$ or $N^{\mathrm{vN}}$). As pointed out above, $N$ is *rigid*, *i.e.*, there are no automorphisms on $N$ other than the identity mapping. It follows from this that any property (conceived as a set) of the elements in $\mathbb{N}$ is invariant in the above sense. Hence, there are $2^N$ invariant properties of natural numbers. However, assuming that our language or arithmetic $\mathcal{L}_{PA}$ is finite, there are at most countably many definable properties of natural numbers.

In order to clarify the relation between invariance and definability in the context of our discussion, we need to move back from Keränen's non-eliminative account to a more neutral understanding of structuralism and translate his claim into our preferred way of thinking about invariance *across* systems, that is, invariance with respect to the isomorphisms between systems. Given the previous distinction between properties of structured systems and properties of the elements of such systems, his assumption can be reformulated in terms of two equivalence claims, namely:

($\alpha$) A property of systems of type $\mathbf{T}$ is invariant under isomorphisms between $\mathbf{T}$-systems iff it is $\mathcal{L}_{\mathbf{T}}$-definable.

($\alpha^*$) A property of elements in a $\mathbf{T}$-system $\mathcal{S}$ is invariant under isomorphisms between $\mathbf{T}$-systems iff it is $\mathcal{L}_{\mathbf{T}}$-definable in $\mathcal{S}$.

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[25] See, for instance, [Hodges, 1997, p. 93].

Do these equivalences hold? The respective right-to-left directions can easily be confirmed. Put differently, the definability account subsumes the invariance account. This is a simple consequence of the isomorphism lemma from elementary model theory, *viz.*, the result that isomorphic systems are semantically equivalent.[26] Thus, for case ($\alpha$), the following holds:

**Observation 1.** *Let* $\mathsf{P}$ *be a property of systems of type* $\mathbf{T}$*. If* $\mathsf{P}$ *is* $\mathcal{L}_{\mathbf{T}}$*-definable, then it is invariant.*

*Proof sketch:* Intuitively, what needs to be shown here is that any class of systems defined in $\mathcal{L}_{\mathbf{T}}$ consists of full isomorphism classes of $\mathbf{T}$s. Let the extension of $\mathsf{P}$ be defined by sentence $\Phi_{\mathsf{P}}$ and let $\mathcal{S} \in \mathbf{T}$ such that $\mathcal{S} \models \Phi_{\mathsf{P}}$. Then, by the isomorphism lemma, for any $X \in \mathbf{T}$: if $X \simeq \mathcal{S}$ we also have $X \models \Phi_{\mathsf{P}}$. A formal proof of this can be given by induction on the complexity of formulas.[27] An analogous argument can be given in support of the claim that $\mathcal{L}$-definable properties of elements in a system $\mathcal{S}$ are invariant under all isomorphic copies of $\mathcal{S}$.

While this direction of the equivalence claims is straightforward, the left-to-right direction is not. Specifically, it is not *generally* the case that invariant properties are also definable in the particular language of the theory in question. To see this, consider some counterexamples against claim ($\alpha$): the property of first-order *PA*-systems of 'being isomorphic to the standard number system $\mathbf{N}$' is not definable in first-order $\mathcal{L}_{PA}$, but is clearly invariant under isomorphisms of *PA*-systems. As is well known, the property becomes characterizable if one works within a second-order language of arithmetic, simply due to the fact that the second-order formulation of *PA* (with the second-order axiom of induction) gives a categorical axiomatization of this number system.[28] One could thus think that the equivalence stated in ($\alpha$) could be restored if a notion of second-order definability were presupposed.[29] However, this is also not the case. One

[26] **Lemma** ('Isomorphism Lemma'): Let $\mathcal{L}$ be a language. Then for all models $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{L}$ and all $\varphi \in \mathcal{L}$: $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \Rightarrow \mathcal{M} \vDash \varphi \Leftrightarrow \mathcal{N} \vDash \varphi$.

[27] A full proof of the isomorphism lemma for first-order languages is given in [Marker, 2000, pp. 13–14]. This proof can easily be generalized to apply also to higher-order languages.

[28] *PA* is categorical if one assumes a standard semantics for the second-order language in which the theory is formulated. See [Shapiro, 1991, §4.2] for a detailed discussion of different semantics for second-order languages as well as for a proof sketch of the categoricity of arithmetic.

[29] It should be noted here that using second-order languages (and higher-order languages more generally) in this context might be seen as problematic since this would qualify certain properties of mathematical objects as structural that we would intuitively not count as such. Consider, for instance, the second-order language for real-number fields. One can easily formulate statements in this language which are true if and only if the continuum hypothesis is true. However, few number theoretists would take such a statement as expressing a structural property of the real-number systems. We would like to thank an anonymous reviewer for drawing our attention to this fact.

can construct examples of invariant properties that fail to be second-order definable. To give one example:

**Observation 2.** *Let* $\mathcal{L}_{PA}^{\vDash_2} := \mathcal{L}_{PA} \cup \{\vDash_2\}$*, where* $\vDash_2$ *is a unary predicate that is intended to apply exactly to the codes of all the second-order validities. Then the property of 'being isomorphic to* $\langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^2 \rangle$*', where* $\mathsf{Val}^2$ *is the set of codes of second-order validities, is not second-order definable, but is invariant under isomorphisms between* $\mathcal{L}_{PA}^{\vDash_2}$*-models.*[30]

*Proof.* By Theorem 41C of [Enderton, 2001, p. 268] the set $\mathsf{Val}^2$ is not definable in $\langle \mathbb{N}^N, 0^N, S^N \rangle$ by any formula of second-order logic. Suppose that we could define the property of being isomorphic to $\langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^2 \rangle$ by a formula $\varphi$, *i.e.*, $\mathcal{M} \cong \langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^2 \rangle$ iff $\mathcal{M} \vDash \varphi$. Then $n \in \mathsf{Val}^2$ iff $\varphi \rightarrow \vDash_2 (\underline{n})$ is valid. To see this, note that if $\mathcal{M} \vDash \vDash_2(\underline{n})$ and $\mathcal{M} \cong \langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^2 \rangle$, then $n \in \mathsf{Val}_2$. And so if we could define the property of being isomorphic to $\langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^2 \rangle$, $\mathsf{Val}^2$ would be definable in $\langle \mathbb{N}^N, 0^N, S^N \rangle$ after all, in contradiction to Enderton's Theorem 41C. So the property of being isomorphic to $\langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^2 \rangle$ is not definable by any formula in second-order logic. □

The latter example suggests that however strong one chooses one's background language to be, there is always a way to construct examples of mathematical properties that are invariant but fail to be definable in that language. An important exception to this, at least for the case of properties of elements in systems, is the limit case where definability is specified relative to an *infinitary* language. Let $\mathcal{L}_{\infty,\infty}$ be a language of *pure* infinitary logic, *i.e.*, an extension of a first-order language that allows formulas with infinitely long sequences of conjunctions, disjunctions, and quantifiers. There are a number of well-known mathematical results — discussed mainly in the debate on invariance-criteria for logical notions — that show that the notions definable in such a language in fact coincide with those meeting certain invariance conditions.[31] The central result in this debate, presented in [McGee, 1996], is based on Tarski's [1986] suggestion of characterizing logical notions in terms of invariance under permutations of a given domain of objects. McGee's theorem states,

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[30] As a consequence of a result of Hintikka [1955], this observation generalizes to higher-order logics beyond second-order logic. Consider the language $\mathcal{L}_{PA}^{\vDash_n} := \mathcal{L}_{PA} \cup \{\vDash_n\}$, where $\vDash_n$ is intended to apply to exactly the codes of the validities of $n^{\text{th}}$-order logic, for $n$ a natural number. Then the property of 'being isomorphic to $\langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^n \rangle$', where $\mathsf{Val}^n$ is the set of codes of second-order validities, is not $n^{\text{th}}$-order definable. The reason is that by Hintikka's result $\mathsf{Val}^n$ is (computably) reducible to $\mathsf{Val}^2$. More specifically, for every sentence $\varphi \in \mathcal{L}_{PA}^{\vDash_n}$ we can find a sentence $\psi \in \mathcal{L}_{PA}^{\vDash_2}$ such that $\varphi$ is valid in $n^{\text{th}}$-order logic iff $\psi$ is valid in second-order logic. Thus, if 'being isomorphic to $\langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^n \rangle$' were definable in $\mathcal{L}_{PA}^{\vDash_n}$, then 'being isomorphic to $\langle \mathbb{N}^N, 0^N, S^N, \mathsf{Val}^2 \rangle$' would be definable in $\mathcal{L}_{PA}^{\vDash_2}$, which we have already observed is not the case. Montague [1965] has shown that Hintikka's result even extends to infinitary-order languages, and thus our observation also applies there.

[31] See, in particular, [McGee, 1996; Bonnay, 2008; Bonnay and Engström, 2015].

roughly, that a logical operation is invariant under all permutations of a given domain if and only it is definable in $\mathcal{L}_{\infty,\infty}$. Note that this result is more general than the invariance and definability conditions discussed in the present paper. Invariance in the context of logical operations means invariance under *all* permutations of a given domain, not invariance under permutations that preserve some additional (mathematical) structure. Similarly, definability means definability in a *pure* logical language without non-logical constants for McGee. In contrast, our focus in the present context is on mathematical properties that are definable in languages with a non-empty signature.

That said, one can easily present a relativized version of McGee's theorem that connects definability in a mathematical language with invariance under the automorphisms of a given model of that language. Let $\mathcal{L}_{\infty,\infty,\mathbf{T}}$ be an infinitary language with a signature of mathematical type $\mathbf{T}$ and let $\mathcal{M}$ be an interpretation of it. In order to keep the following discussion simple, we assume here that $\mathcal{M}$ is a purely relational system of the form $\langle D, R_1, \ldots, R_n \rangle$ with $R_1, \ldots, R_n$ first-order relations (of a given arity) on an infinite domain $D$.

**Observation 3.** *For all systems $\mathcal{M}$ of type $\mathbf{T}$, the extension of a property $\mathsf{P}$ in $\mathcal{M}$ is invariant under all automorphisms of $\mathcal{M}$; in symbols, $\mathsf{P} \in Inv(Aut(\mathcal{M}))$, iff the extension of $\mathsf{P}$ in $\mathcal{M}$ is definable in $\mathcal{L}_{\infty,\infty,\mathbf{T}}$.*[32]

*Proof.* ($\Leftarrow$) Assume that the extension of $\mathsf{P}$ in $\mathcal{M}$ is definable by a formula $\varphi(x) \in \mathcal{L}_{\infty,\infty,\mathbf{T}}$, *i.e.*, $\{d \in D \mid \mathsf{P}_{\mathcal{M}}(d)\} = \{d \in D \mid \mathcal{M} \vDash \varphi(\underline{d})\}$. Let $f$ be an automorphism of $\mathcal{M}$. By definition, for any $R^{\mathcal{M}}$ (of arity $n$) in $\mathcal{M}$, we have: $(d_1, \ldots, d_n) \in R^{\mathcal{M}} \Leftrightarrow (f(d_1), \ldots, f(d_n)) \in R^{\mathcal{M}}$. One can show by straightforward induction on the complexity of formulas that $\mathcal{M} \models \varphi(\underline{d})$ iff $\mathcal{M} \models \varphi(\underline{f(d)})$. Hence, $\mathsf{P}_{\mathcal{M}}(d)$ iff $\mathsf{P}_{\mathcal{M}}(f(d))$, and thus $\mathsf{P} \in Inv(Aut(\mathcal{M}))$.

($\Rightarrow$) Let $\mathsf{P} \in Inv(Aut(\mathcal{M}))$. We construct a sentence $\varphi \in \mathcal{L}_{\infty,\infty,\mathbf{T}}$ that defines it. Take an enumeration $I \to D$ of the elements in $D$, and pick a set of variables $\mathcal{V}$ of cardinality $|D|$ enumerated by $I \to \mathcal{V}$. Let $J = \{j \in I \mid \mathsf{P}(d_j)\}$ be the set of indices in $I$ that enumerate the members of the extension of $\mathsf{P}$ in $\mathcal{M}$. Next, for every $n$-ary relation $R^{\mathcal{M}}$ and every $K \subseteq I$ with $|K| = n$, let $\delta_K^{R^{\mathcal{M}}}$ be $R(x_{k_1}, \ldots, x_{k_n})$ if $(d_{k_1}, \ldots, d_{k_n}) \in R^{\mathcal{M}}$ and $\neg R(x_{k_1}, \ldots, x_{k_n})$ if $(d_{k_1}, \ldots, d_{k_n}) \notin R^{\mathcal{M}}$. Then the description $\Delta_{R^{\mathcal{M}}}$ of $R^{\mathcal{M}}$ in $\mathcal{L}_{\infty,\infty,\mathbf{T}}$ is the conjunction:

$$\Delta_{R^{\mathcal{M}}} = \bigwedge_{K \subseteq I} \delta_K^{R^{\mathcal{M}}}.$$

Now, we can let $\varphi$ be:

$$(\exists x_i)_{i \in I}\Big( \bigwedge_{i,j \in I, i \neq j} (x_i \neq x_j) \wedge \forall y \bigvee_{i \in I} (y = x_i) \wedge \bigwedge_{R \in \mathcal{R}} \Delta_{R^{\mathcal{M}}} \wedge \bigvee_{j \in J} x = x_j\Big).$$

[32] A version of this result with second-order relations is presented in [Bonnay and Engström, 2015]. Our proof of this result (for first-order relations) is based on the proof in [Rogers, 1967].

It remains to show that $\varphi$ defines the extension of $\mathsf{P}$ in $\mathcal{M}$, *i.e.*, $\mathsf{P}(d)$ iff $\mathcal{M} \models \varphi(\underline{d})$:

From left to right, let $d$ be in the extension of $\mathsf{P}$ in $\mathcal{M}$. Then $d = d_j$, for some $j \in J$. Now consider any assignment $\sigma$ in $\mathcal{M}$. Then simply pick some assignment $\sigma'$, such that $\sigma'(x_i) = d_i$, which will be a $\mathcal{V}$-variant of $\sigma$. Then the formula:

$$\bigwedge_{i,j \in I, i \neq j} (x_i \neq x_j) \wedge \forall y \bigvee_{i \in I} (y = x_i) \wedge \bigwedge_{R \in \mathcal{R}} \Delta_{R^{\mathcal{M}}} \wedge \bigvee_{j \in J} \underline{d} = x_j \qquad (*)$$

is clearly satisfied relative to $\sigma'$. Hence $\varphi$ is satisfied relative to $\sigma$, which is what we needed to show.

From right to left, let $\sigma$ be an assignment such that $\varphi(\underline{d})$ is satisfied in $\mathcal{M}$ relative to $\sigma$. Hence there is a $\mathcal{V}$-variant $\sigma'$ of $\sigma$, such that $(*)$ is satisfied relative to $\sigma'$. Now consider the values $\sigma'(x_i) \in D$ of the variables $x_i \in \mathcal{V}$ for $i \in I$. Since $\sigma'$ satisfies the first and second conjuncts of $(*)$, we know that $D = \{\sigma'(x_i) \mid i \in I\}$. Thus, we can find a permutation $f : D \to D$ such that $f(\sigma'(x_i)) = d_i$ for $i \in I$. Now note that since $\sigma'$ satisfies the third conjunct of $(*)$, $f \circ \sigma'$ is an automorphism of $\mathcal{M}$ (simply inspect the definition of $\delta_K^{R^{\mathcal{M}}}$). Finally, since $\sigma'$ satisfies the fourth conjunct of $(*)$, we know that $d = \sigma'(x_j)$ for some $j \in J$. Then for this $j \in J$ $f(\sigma'(x_j)) = d$ is in the extension of $\mathsf{P}$ in $\mathcal{M}$. But since $f \circ \sigma'$ is an automorphism and $\mathsf{P}$ is invariant under automorphisms by assumption, this means that $d$ is in the extension of $\mathsf{P}$ in $\mathcal{M}$, which is what we needed to show. □

This result shows that if a sufficiently strong infinitary language is adopted, invariant properties always turn out to be definable. Notice, however, that the invariance condition used here, namely invariance under automorphisms of a given system, has been rejected by us as a proper way to explicate the notion of structural properties. Thus, as it stands, Observation 3 is not directly relevant to our present investigation.

Nevertheless, there exists an interesting generalization of McGee's theorem that turns out to be applicable to the present discussion. The so-called Tarski-Sher thesis, also formulated in the debate on the nature of logical constants, states that an operation is logical if and only if it is invariant under bijections across domains. Thus, in contrast to Tarski's original thesis, logicality is defined here not in terms of invariance relative to a given domain, but in terms of invariance *across* domains. Given this approach, McGee [1996] has formulated an interesting corollary of his result about logical notions (understood now as operations *across* domains) that are invariant under bijections. The theorem states that an operation $\mathsf{P}$ is invariant under all bijections between domains iff for each cardinal $\kappa \neq 0$ there is a formula $\varphi_\kappa \in \mathcal{L}_{\infty,\infty}$ which describes the action of $\mathsf{P}$ on domains of cardinality $\kappa$.

Applied to the present discussion of mathematical properties, this result can be easily transformed into the following relativized result. Let $\mathsf{P}$ be a property of elements in systems of mathematical type $\mathbf{T}$ and let language $\mathcal{L}_{\infty,\infty,\mathbf{T}}$ be specified as above. Let us say that the extension of $\mathsf{P}$ is invariant under isomorphisms of $\mathbf{T}$-systems iff for all $\mathbf{T}$-systems $\mathcal{S}, \mathcal{T}$, all isomorphisms $\lambda : \mathcal{S} \to \mathcal{T}$, and all elements $d \in D^{\mathcal{S}}$, $\mathsf{P}_{\mathcal{S}}(d) \Rightarrow \mathsf{P}_{\mathcal{T}}(\lambda(d))$. We can then show the following:

**Observation 4.** *The extension of* $\mathsf{P}$ *is invariant under isomorphisms of* $\mathbf{T}$*-systems iff for each equivalence class* $\kappa = [\mathcal{M}]_{\simeq}$ *of isomorphic* $\mathbf{T}$*-systems there is a formula* $\varphi_\kappa \in \mathcal{L}_{\infty,\infty,\mathbf{T}}$ *which defines the extensions of* $\mathsf{P}$ *in all systems* $\mathcal{M} \in \kappa$.

*Proof.* ($\Leftarrow$) Assume that the extension of $\mathsf{P}$ is definable in all systems in all $\kappa = [\mathcal{M}]_{\simeq}$ by formula $\varphi_\kappa(x)$. Let $\mathcal{S}, \mathcal{T}$ be two $\mathbf{T}$-systems. If they are not isomorphic, then the claim trivially holds; so assume without loss of generality that $\mathcal{S} \simeq \mathcal{T}$ and $\lambda : \mathcal{S} \to \mathcal{T}$ is an isomorphism between the two. Since the two systems are isomorphic, we know that $[\mathcal{S}]_{\simeq} = [\mathcal{T}]_{\simeq}$, and so the extension of $\mathsf{P}$ in both systems is defined by the same formula $\varphi_{[\mathcal{S}]_{\simeq}}$. It then follows by a simple inductive argument on the complexity of $\varphi_{[\mathcal{S}]_{\simeq}}$ that for all $d \in D^{\mathcal{S}}$, $\mathsf{P}_{\mathcal{S}}(d) \Rightarrow \mathsf{P}_{\mathcal{T}}(\lambda(d))$, since $\{d \in D^{\mathcal{S}} \mid \mathsf{P}_{\mathcal{S}}(d)\} = \{d \in D^{\mathcal{S}} \mid \mathcal{S} \vDash \varphi_{[\mathcal{S}]_{\simeq}}(\underline{d})\}$ and $\{d \in D^{\mathcal{T}} \mid \mathsf{P}_{\mathcal{T}}(d)\} = \{x \in D^{\mathcal{T}} \mid \mathcal{T} \vDash \varphi_{[\mathcal{S}]_{\simeq}}(\underline{d})\}$ by assumption.

($\Rightarrow$) Assume that the extension of $\mathsf{P}$ is invariant under isomorphisms of $\mathbf{T}$-systems and let $\kappa = [\mathcal{M}]_{\simeq}$ be some equivalence class of isomorphic $\mathbf{T}$-systems. Consider the formula

$$(\exists x_i)_{i\in I}(\bigwedge_{i,j\in I, i\neq j}(x_i \neq x_j) \wedge \forall y \bigvee_{i\in I}(y = x_i) \wedge \bigwedge_{R\in\mathcal{R}} \Delta_{R^{\mathcal{M}}} \wedge \bigvee_{j\in J} x = x_j)$$

from the proof of Observation 3 now defined for the representative $\mathcal{M}$ of $\kappa$. We can take this formula to be $\varphi_\kappa$. To establish this, we need to show that in all $\mathcal{N} \in \kappa$, $\mathsf{P}_{\mathcal{N}}(d)$ iff $\mathcal{N} \vDash \varphi_\kappa(\underline{d})$. The proof of this fact goes along the same lines as the proof of Observation 3.

For the left-to-right direction, assume that $\mathsf{P}_{\mathcal{N}}(d)$ and let $\lambda : \mathcal{N} \to \mathcal{M}$ be an isomorphism. Since $\mathsf{P}$ is invariant under isomorphisms of $\mathbf{T}$-systems, this means that $\mathsf{P}_{\mathcal{M}}(\lambda(d))$ and thus $\lambda(d) = d_j$ for some $j \in J$. By the same argument as in the proof of Observation 3, we get that $\mathcal{M} \vDash \varphi_\kappa(\underline{\lambda(d)})$. But since $\lambda$ is an isomorphism, it follows by the isomorphism lemma of model theory that $\mathcal{N} \vDash \varphi_\kappa(\underline{d})$, as desired.

For the right-to-left direction, assume that $\mathcal{N} \vDash \varphi_\kappa(\underline{d})$. Hence, again by the isomorphism lemma, we get that $\mathcal{M} \vDash \varphi_\kappa(\underline{\lambda(d)})$ and by the same argument as in the proof of Observation 3 that $\mathsf{P}_{\mathcal{M}}(\lambda(d))$. But since $\mathsf{P}$ is assumed to be invariant under isomorphisms of $\mathbf{T}$-systems, we get that $\mathsf{P}_{\mathcal{N}}(d)$ as desired. □

This adaptation of McGee's second theorem shows that the local extension of a property is always definable if the property is invariant under isomorphisms. More generally, these results indicate that the invariance-based and definability-based explications of structural properties tend to converge if one increases the logical strength of the language in use. Moreover, they determine the same class of properties of a given mathematical type if we choose an infinitary language as our mathematical language. Leaving aside such limit cases, however, the examples presented above show that the equivalence claims ($\alpha$) and ($\alpha^*$) are not generally true. In particular, it is not the case that all invariant properties of mathematical objects are also definable in the languages in which these objects are usually described. It follows that the two explications suggested here do not determine the same pre-theoretical notion of structural properties. Where does this leave us in our philosophical assessment of the paper's main question?

More generally, two different philosophical morals can be drawn from the above observations. The first one is to uphold the view that there exists a *unique* pre-theoretical notion of structural properties that can be captured formally by one of the two accounts presented above. What needs to be considered then is which one of the two explications is more adequate or, in Carnap's terminology, which one bears more similarity to the pre-theoretic explicandum. The problem this approach faces is that neither of the two explications fully captures our informal understanding of the notion. Specifically, the invariance account tends to *overgenerate*, that is, it carves out more properties as structural than we would do so intuitively. In contrast, the definability account (specified relative to a particular language) tends to *undergenerate*: It fails to specify properties as structural that we would intuitively characterize in this way. Starting with the first problem, we can consider two types of properties that qualify as structural according to the invariance-based definition, but that do not plausibly qualify as structural pre-theoretically. We will dub them *propositional* and *parasitic* properties. Roughly, a *propositional property* is a property such that there is a proposition that, for everything, having the property is equivalent to the proposition's being true, *i.e.*, $Prop(\mathsf{P})$ iff $\exists p \forall x(\mathsf{P}(x) \leftrightarrow p)$.

Propositional properties are problematic for the invariance-based explication. Since for every proposition $p$ it is a logical truth that $p \rightarrow p$, it follows immediately that if $\mathsf{P}$ is a propositional property, then $\mathsf{P}$ is invariant under every equivalence relation, *i.e.*, about every subject matter. Hence, in particular, any propositional property counts as structural under Explication 1. But it clearly seems unnatural to take all such properties as structural given that they express no information about the structural composition of particular objects or their relations to other objects in a system. To illustrate this point, consider the propositional property of groups *such that a group of order two exists*. By the previous reasoning this property is a structural property of groups. But pre-theoretically we should only say of groups of order two that they have the property of *being such that a group of order two exists* in virtue of *their* internal structure and not the structure of something else. Hence the problem.

In turn, we call a property 'parasitic' if its formulation is based on the isomorphism relation for a given mathematical type $\mathbf{T}$. It turns out that no matter how we define $\mathsf{Iso}_{\mathbf{T}}(x, y)$, the property $\lambda x(\exists y \neg \mathsf{Iso}_{\mathbf{T}}(x, y))$ is always trivially invariant under the isomorphisms in $\mathbf{T}$. However, pre-theoretically $\lambda x(\exists y \neg \mathsf{Iso}_{\mathbf{T}}(x, y))$ should not count as a structural property of the $\mathbf{T}$s as it says that there is something with a different structure and a system has that property not in virtue of its own intrinsic structure but in virtue of this external fact. Parasitic and propositional properties of this sort can be viewed as philosophical counterexamples against the view that the invariance-based explications fully capture our pre-theoretical understanding of structural properties. What is typical of these counterexamples is that the properties are invariant, however, not in virtue of the *internal* structural composition of the systems they hold of. The properties in question have as their extension either the full type $\mathbf{T}$ or the empty set $\emptyset$. Thus, statements referring to them are non-informative about the structure of the objects in question.

Turning to the undergeneration problem, it was already shown above that the definability account fails to provide an intuitively satisfying demarcation between structural and nonstructural properties. This has to do with the fact that the specification of structural properties based on Explications 3 and 4 is strongly *dependent* on the choice of a particular logical background language. In particular, a property may fail to be structural according to the definability approach purely because of the limitations of the *logical resources* of the formal language in use. Depending on the expressive strength of a particular language, some intuitively structural properties can turn out to be indefinable: 'being isomorphic to $N = \langle \mathbb{N}^N, 0^N, S^N \rangle$' was a case in point here for first-order languages of arithmetic. This language relativity makes it difficult to consider Explications 3 and 4 as the most adequate ways to formalize structural properties in mathematics.[33]

The alternative and in our view more promising approach is to take a *Carnapian* or tolerant view on this matter. This is to embrace the fact that there are different ways to formalize our informal understanding of the notion. In contrast to the above view, the invariance and the definability accounts should thus be seen as two equally legitimate ways to make precise what we mean by structural properties of mathematical objects in informal discourse. Moreover, there are, as we saw, different accounts of definable properties corresponding to the spectrum of possible logical background languages. The choice of one formalization over the other should thus be based on purely practical considerations in a particular theoretical context and *not* on a general ideal of correctness or truth. For instance, for certain mathematical or foundational questions, the invariance-based approach seems preferable because it provides us with a very comprehensive account of structural properties. In other contexts, the definability-based approach might seem more useful, in particular in cases where one is looking for a more tractable or constructive criterion for structurality in mathematics.

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[33] A related form of language relativity has been discussed in debates on mathematical structuralism. See, in particular, Resnik [1997] on this point. He describes in detail the 'structural relativity' underlying his non-eliminative theory of patterns or structures there. This is the fact that 'the structures we can discern and describe are a function of the background devices we have available for depicting structures' [1997, p. 250].

## 6. CONCLUSION

The paper presented several explications of the notion of structural properties of mathematical systems as well as of the elements of such systems. According to the invariance-based approach, properties are structural if they remain invariant under all isomorphic transformations. According to the definability-based approach, structural properties are those definable in the language of a given mathematical theory. It was shown that the two ways of explicating structural properties usually do not determine the same pre-theoretical notion.

In what sense are the results given here relevant for present discussions in philosophy of mathematics and for mathematical structuralism in particular? As mentioned in the introduction, the aim in the paper was not to argue for a particular version of mathematical structuralism (such as '*ante rem*' or 'universal' structuralism). Rather, it was to give a general logical analysis of the notion of structural properties that allows one to see how the notion could be understood (in a precise sense) in different philosophical debates. Such debates concern, for instance, the ontological dependence between places and pure structures in which they occur [Linnebo, 2008], the identity of structurally indiscernible places [Leitgeb and Ladyman, 2008; Keränen, 2001; Shapiro, 2008] in non-eliminative structuralism, or the formulation of consistent 'structural' abstraction principles [Linnebo and Pettigrew, 2014]. In each of these specialized discussions, structural properties play a central role. We hope that the present attempt to precisify the notion in terms of invariance and definability conditions will be useful for future work on these topics.

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